PARTIAL DIFFERENTIAL EQUATIONS MA 3132 SOLUTIONS OF PROBLEMS IN LECTURE NOTES

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CHAPTER 1

1 Introduction and Applications

1.1 Basic Concepts and Definitions

Problems

1. Give the order of each of the following PDEs

a. $u_{xx} + u_{yy} = 0$
b. $u_{xxx} + u_{xy} + a(x)u_y + \log u = f(x,\, y)$
c. $u_{xxx} + u_{xyyy} + a(x)u_{xxy} + u^2 = f(x,\, y)$
d. $u\, u_{xx} + u_{yy}^2 + e^u = 0$
e. $u_x + cu_y = d$

2. Show that

$$u(x,\, t) = \cos(x - ct)$$

is a solution of

$$u_t + cu_x = 0$$

3. Which of the following PDEs is linear? quasilinear? nonlinear? If it is linear, state whether it is homogeneous or not.

a. $u_{xx} + u_{yy} - 2u = x^2$
b. $u_{xy} = u$
c. $u\, u_x + x\, u_y = 0$
d. $u_x^2 + \log u = 2xy$
e. $u_{xx} - 2u_{xy} + u_{yy} = \cos x$
f. $u_x(1 + u_y) = u_{xx}$
g. $(\sin u_x)u_x + u_y = e^x$
h. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$
i. $u_x + u_x u_y - u_{xy} = 0$

4. Find the general solution of

$$u_{xy} + u_y = 0$$

(Hint: Let $v = u_y$)

5. Show that

$$u = F(xy) + x\, G(\frac{y}{x})$$

is the general solution of

$$x^2 u_{xx} - y^2 u_{yy} = 0$$

1. a. Second order
   b. Third order
   c. Fourth order
   d. Second order
   e. First order

2. $u = \cos(x - ct)$

$$u_t = -c \cdot (-\sin(x - ct)) = c\sin(x - ct)$$
$$u_x = 1 \cdot (-\sin(x - ct)) = -\sin(x - ct)$$
$$\Rightarrow u_t + cu_x = c\sin(x - ct) - c\sin(x - ct) = 0.$$

3. a. Linear, inhomogeneous
   b. Linear, homogeneous
   c. Quasilinear, homogeneous
   d. Nonlinear, inhomogeneous
   e. Linear, inhomogeneous
   f. Quasilinear, homogeneous
   g. Nonlinear, inhomogeneous
   h. Linear, homogeneous
   i. Quasilinear, homogeneous

4.

$$u_{xy} + u_y = 0$$

Let $v = u_y$ then the equation becomes

$$v_x + v = 0$$

For fixed y, this is a separable ODE

$$\frac{dv}{v} = -dx$$

$$\ln v = -x + C(y)$$

$$v = K(y)\, e^{-x}$$

In terms of the original variable u we have

$$u_y = K(y)\, e^{-x}$$

$$u = e^{-x}\, q(y) + p(x)$$

You can check your answer by substituting this solution back in the PDE.

5.

$$u = F(xy) + x\,G\left(\frac{y}{x}\right)$$

$$u_x = y\,F'(xy) + G\left(\frac{y}{x}\right) + x\left(-\frac{y}{x^2}\right)G'\left(\frac{y}{x}\right)$$

$$u_{xx} = y^2\,F''(xy) + \left(-\frac{y}{x^2}\right)G'\left(\frac{y}{x}\right) - \frac{y}{x}\left(-\frac{y}{x^2}\right)G''\left(\frac{y}{x}\right) + \left(\frac{y}{x^2}\right)G'\left(\frac{y}{x}\right)$$

$$u_{xx} = y^2\,F''(xy) + \frac{y^2}{x^3}\,G''\left(\frac{y}{x}\right)$$

$$u_y = x\,F'(xy) + x\,\frac{1}{x}G'\left(\frac{y}{x}\right)$$

$$u_{yy} = x^2\,F''(xy) + \frac{1}{x}G''\left(\frac{y}{x}\right)$$

$$x^2u_{xx} - y^2u_{yy} = x^2\left(y^2\,F'' + \frac{y^2}{x^3}\,G''\right) - y^2\left(x^2\,F'' + \frac{1}{x}\,G''\right)$$

Expanding one finds that the first and third terms cancel out and the second and last terms cancel out and thus we get zero.

1.2 Applications

1.3 Conduction of Heat in a Rod

1.4 Boundary Conditions

Problems

1. Suppose the initial temperature of the rod was

$$u(x,\,0) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 2(1-x) & 1/2 \le x \le 1 \end{cases}$$

and the boundary conditions were

$$u(0,\,t) = u(1,\,t) = 0\ ,$$

what would be the behavior of the rod's temperature for later time?

2. Suppose the rod has a constant internal heat source, so that the equation describing the heat conduction is

$$u_t = k u_{xx} + Q, \qquad 0 < x < 1\ .$$

Suppose we fix the temperature at the boundaries

$$\begin{aligned} u(0,\,t) &= 0 \\ u(1,\,t) &= 1\ . \end{aligned}$$

What is the steady state temperature of the rod? (Hint: set $u_t = 0$.)

3. Derive the heat equation for a rod with thermal conductivity $K(x)$.

4. Transform the equation

$$u_t = k(u_{xx} + u_{yy})$$

to polar coordinates and specialize the resulting equation to the case where the function u does NOT depend on θ. (Hint: $r = \sqrt{x^2 + y^2}$, $\tan\theta = y/x$)

5. Determine the steady state temperature for a one-dimensional rod with constant thermal properties and

a. $Q = 0$, $\quad u(0) = 1$, $\quad u(L) = 0$
b. $Q = 0$, $\quad u_x(0) = 0$, $\quad u(L) = 1$
c. $Q = 0$, $\quad u(0) = 1$, $\quad u_x(L) = \varphi$

d. $\frac{Q}{k} = x^2$, $\quad u(0) = 1$, $\quad u_x(L) = 0$

e. $Q = 0$, $\quad u(0) = 1$, $\quad u_x(L) + u(L) = 0$

1. Since the temperature at both ends is zero (boundary conditions), the temperature of the rod will drop until it is zero everywhere.

2.

$$k\, u_{xx} \,+\, Q \,=\, 0$$
$$u(0.t) = 0$$
$$u(1,t) = 1$$

$\Rightarrow \qquad u_{xx} \,=\, -\frac{Q}{k}$

Integrate with respect to x

$$u_x \,=\, -\frac{Q}{k}\,x \,+\, A$$

Integrate again

$$u \,=\, -\frac{Q}{k}\,\frac{x^2}{2} \,+\, A\,x \,+\, B$$

Using the first boundary condition $u(0) = 0$ we get $B \,=\, 0$. The other boundary condition will yield

$$-\frac{Q}{k}\,\frac{1}{2} \,+\, A \,=\, 1$$

$$\Rightarrow \qquad A \,=\, \frac{Q}{2k} \,+\, 1$$

$$\Rightarrow \qquad u(x) \,=\, \left(1 \,+\, \frac{Q}{2k}\right)\, x \,-\, \frac{Q}{2k}\,x^2$$

3. Follow class notes.

4.

$$r = \left(x^2 + y^2\right)^{\frac{1}{2}}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$r_x = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2x \ = \ \frac{x}{\sqrt{x^2 + y^2}}$$

$$\theta_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\ \left(-\frac{y}{x^2}\right) \ = \ -\frac{y}{x^2 + y^2}$$

$$r_y = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2y \ = \ \frac{y}{\sqrt{x^2 + y^2}}$$

$$\theta_y = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\ \left(\frac{1}{x}\right) \ = \ \frac{x}{x^2 + y^2}$$

$$u_x = u_r r_x \ + \ u_\theta \theta_x \ = \ \frac{x}{\sqrt{x^2 + y^2}}\, u_r \ - \ \frac{y}{x^2 + y^2}\, u_\theta$$

$$u_y = u_r r_y \ + \ u_\theta \theta_y \ = \ \frac{y}{\sqrt{x^2 + y^2}}\, u_r \ + \ \frac{x}{x^2 + y^2}\, u_\theta$$

$$u_{xx} \ = \ \left(\frac{x}{\sqrt{x^2 + y^2}}\right)_x u_r \ + \ \frac{x}{\sqrt{x^2 + y^2}}\ (u_r)_x \ - \ \left(\frac{y}{x^2 + y^2}\right)_x u_\theta \ - \ \frac{y}{x^2 + y^2}\ (u_\theta)_x$$

$$u_{xx} \ = \ \frac{\sqrt{x^2 + y^2} \ - \ x\frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2x}{x^2 + y^2}\, u_r \ + \ \frac{x}{\sqrt{x^2 + y^2}}\left[\frac{x}{\sqrt{x^2 + y^2}}\, u_{rr} \ - \ \frac{y}{x^2 + y^2}\, u_{r\theta}\right]$$
$$- \ \frac{-2xy}{\left(x^2 + y^2\right)^2}\, u_\theta \ - \ \frac{y}{x^2 + y^2}\ \left[\frac{x}{\sqrt{x^2 + y^2}}\, u_{r\theta} \ - \ \frac{y}{x^2 + y^2}\, u_{\theta\theta}\right]$$

$$u_{xx} \ = \ \frac{x^2}{x^2 + y^2}\, u_{rr} \ - \ \frac{2xy}{\left(x^2 + y^2\right)^{\frac{3}{2}}}\, u_{r\theta} \ + \ \frac{y^2}{\left(x^2 + y^2\right)^2}\, u_{\theta\theta} \ + \ \frac{y^2}{\left(x^2 + y^2\right)^{\frac{3}{2}}}\, u_r \ + \ \frac{2xy}{\left(x^2 + y^2\right)^2}\, u_\theta$$

$$u_{yy} \ = \ \left(\frac{y}{\sqrt{x^2 + y^2}}\right)_y u_r \ + \ \frac{y}{\sqrt{x^2 + y^2}}\ (u_r)_y \ + \ \left(\frac{x}{x^2 + y^2}\right)_y u_\theta \ + \ \frac{x}{x^2 + y^2}\ (u_\theta)_y$$

$$u_{yy} \ = \ \frac{\sqrt{x^2 + y^2} \ - \ y\frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2y}{x^2 + y^2}\, u_r \ + \ \frac{y}{\sqrt{x^2 + y^2}}\left[\frac{y}{\sqrt{x^2 + y^2}}\, u_{rr} \ + \ \frac{x}{x^2 + y^2}\, u_{r\theta}\right]$$
$$+ \ \frac{-2xy}{\left(x^2 + y^2\right)^2}\, u_\theta \ + \ \frac{x}{x^2 + y^2}\ \left[\frac{y}{\sqrt{x^2 + y^2}}\, u_{r\theta} \ + \ \frac{x}{x^2 + y^2}\, u_{\theta\theta}\right]$$

$$u_{yy} \ = \ \frac{y^2}{x^2 + y^2}\, u_{rr} \ + \ \frac{2xy}{\left(x^2 + y^2\right)^{\frac{3}{2}}}\, u_{r\theta} \ + \ \frac{x^2}{\left(x^2 + y^2\right)^2}\, u_{\theta\theta} \ + \ \frac{x^2}{\left(x^2 + y^2\right)^{\frac{3}{2}}}\, u_r \ - \ \frac{2xy}{\left(x^2 + y^2\right)^2}\, u_\theta$$

$\Rightarrow u_{xx} + u_{yy} = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r$

$$\boxed{u_t = k\left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}\right)}$$

In the case u is independent of θ:

$$\boxed{u_t = k\left(u_{rr} + \frac{1}{r} u_r\right)}$$

5. $k\,u_{xx} + Q = 0$

a. $\quad k\,u_{xx} = 0$

Integrate twice with respect to x

$$u(x) = Ax + B$$

Use the boundary conditions

$$u(0) = 1 \qquad \text{implies } B = 1$$

$$u(L) = 0 \qquad \text{implies } AL + B = 0 \qquad \text{that is } A = -\frac{1}{L}$$

Therefore

$\boxed{u(x) = -\frac{x}{L} + 1}$

b. $\quad k\,u_{xx} = 0$

Integrate twice with respect to x as in the previous case

$$u(x) = Ax + B$$

Use the boundary conditions

$$u_x(0) = 0 \qquad \text{implies } A = 0$$

$$u(L) = 1 \qquad \text{implies } AL + B = 1 \qquad \text{that is } B = 1$$

Therefore

$\boxed{u(x) = 1}$

c. $\quad k\,u_{xx} = 0$

Integrate twice with respect to x as in the previous case

$$u(x) = Ax + B$$

Use the boundary conditions

$$u(0) = 1 \qquad \text{implies } B = 1$$

$$u_x(L) = \varphi \qquad \text{implies } A = \varphi$$

Therefore

$\boxed{u(x) = \varphi\,x + 1}$

d. $\quad k\, u_{xx} \,+\, Q \,=\, 0$

$u_{xx} \,=\, -\,\frac{Q}{k} \,=\, -\,x^2$

Integrate with respect to x we get

$$u_x(x) \,=\, -\,\frac{1}{3}\,x^3 \,+\, A$$

Use the boundary condition

$$u_x(L) \,=\, 0 \qquad \text{implies} \; -\,\frac{1}{3}\,L^3 \,+\, A \,=\, 0 \qquad \text{that is } A \,=\, \frac{1}{3}\,L^3$$

Integrating again with respect to x

$$u \,=\, -\,\frac{x^4}{12} \,+\, \frac{1}{3}\,L^3\,x \,+\, B$$

Use the second boundary condition

$$u(0) \,=\, 1 \qquad \text{implies } B \,=\, 1$$

Therefore

$$\boxed{u(x) \,=\, -\frac{x^4}{12} \,+\, \frac{L^3}{3}\,x \,+\, 1}$$

e. $\quad k\, u_{xx} \,=\, 0$

Integrate twice with respect to x as in the previous case

$$u(x) \,=\, Ax \,+\, B$$

Use the boundary conditions

$$u(0) \,=\, 1 \qquad \text{implies } B \,=\, 1$$

$$u_x(L) \,+\, u(L) = 0 \qquad \text{implies } A \,+\, (AL \,+\, 1) \,=\, 0 \qquad \text{that is } A \,=\, -\,\frac{1}{L+1}$$

Therefore

$$\boxed{u(x) \,=\, -\,\frac{1}{L+1}\,x \,+\, 1}$$

1.5 A Vibrating String

Problems

1. Derive the telegraph equation

$$u_{tt} + au_t + bu = c^2 u_{xx}$$

by considering the vibration of a string under a damping force proportional to the velocity and a restoring force proportional to the displacement.

2. Use Kirchoff's law to show that the current and potential in a wire satisfy

$$\begin{aligned} i_x + C\, v_t + Gv &= 0 \\ v_x + L\, i_t + Ri &= 0 \end{aligned}$$

where $i =$ current, $v = L =$ inductance potential, $C =$ capacitance, $G =$ leakage conductance, $R =$ resistance,

b. Show how to get the one dimensional wave equations for i and v from the above.

1. Follow class notes.

a, b are the proportionality constants for the forces mentioned in the problem.

2. a. Check any physics book on Kirchoff's law.

b. Differentiate the first equation with respect to t and the second with respect to x

$$i_{xt} + C\, v_{tt} + G\, v_t = 0$$

$$v_{xx} + L\, i_{tx} + R\, i_x = 0$$

Solve the first for i_{xt} and substitute in the second

$$i_{xt} = -C\, v_{tt} - G\, v_t$$

$$\Rightarrow v_{xx} - CL\, v_{tt} - GL\, v_t + R\, i_x = 0$$

i_x can be solved for from the original first equation

$$i_x = -C\, v_t - G\, v$$

$$\Rightarrow v_{xx} - CL\, v_{tt} - GL\, v_t - RC\, v_t - RG\, v = 0$$

Or

$$v_{tt} + \left(\frac{G}{C} + \frac{R}{L}\right) v_t + \frac{RG}{CL}\, v = \frac{1}{CL}\, v_{xx}$$

which is the telegraph equation.

In a similar fashion, one can get the equation for i.

CHAPTER 2

2 Classification and Characteristics

2.1 Classification of Linear Second Order PDEs

Problems

1. Classify each of the following as hyperbolic, parabolic or elliptic at every point (x, y) of the domain

a. $x\,u_{xx} + u_{yy} = x^2$
b. $x^2\,u_{xx} - 2xy\,u_{xy} + y^2 u_{yy} = e^x$
c. $e^x u_{xx} + e^y u_{yy} = u$
d. $u_{xx} + u_{xy} - x u_{yy} = 0$ in the left half plane $(x \le 0)$
e. $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} + xy u_x + y^2 u_y = 0$
f. $u_{xx} + x u_{yy} = 0$ (Tricomi equation)

2. Classify each of the following constant coefficient equations

a. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$
b. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$
c. $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$
d. $u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$
e. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$
f. $u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$

3. Use any symbolic manipulator (e.g. MACSYMA or MATHEMATICA) to prove (2.1.19). This means that a transformation does NOT change the type of the PDE.

1a. $A = x$ $B = 0$ $C = 1$ $\triangle = -4x$

hyperbolic for $x < 0$

parabolic $x = 0$

elliptic $x > 0$

b. $A = x^2$ $B = 2xy$ $C = y^2$ $\triangle = 0$ parabolic

c. $A = e^x$ $B = 0$ $C = e^y$ $\triangle = -4e^x e^y$ elliptic

d. $A = 1$ $B = 1$ $C = -x$ $\triangle = 1 + 4x$

hyperbolic $0 \geq x > -\frac{1}{4}$

parabolic $x = -\frac{1}{4}$

elliptic $x < -\frac{1}{4}$

e. $A = x^2$ $B = 2xy$ $C = y^2$ $\triangle = 0$ parabolic

f. $A = 1$ $B = 0$ $C = x$ $\triangle = -4x$

hyperbolic $x < 0$

parabolic $x = 0$

elliptic $x > 0$

2.

	A	B	C	Discriminant	
a.	4	5	1	25 - 16 $>$ 0	hyperbolic
b.	1	1	1	1 - 4 $<$ 0	elliptic
c.	3	10	3	100 - 36 $>$ 0	hyperbolic
d.	1	2	3	4 - 12 $<$ 0	elliptic
e.	2	-4	2	16 - 16 $=$ 0	parabolic
f.	1	5	4	25 - 16 $>$ 0	hyperbolic

3. We substitute for A^*, B^*, C^* given by (2.1.12)-(2.1.14) in Δ^*.

$$
\begin{aligned}
\Delta^* &= (B^*)^2 - 4A^*C^* \\
&= \left[2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y\right]^2 - \\
&\quad 4\left[A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2\right]\left[A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2\right] \\
&= 4A^2\xi_x^2\eta_x^2 + 4A\xi_x\eta_x B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 8A\xi_x\eta_x C\xi_y\eta_y \\
&+ B^2\left(\xi_x\eta_y + \xi_y\eta_x\right)^2 + 4B\left(\xi_x\eta_y + \xi_y\eta_x\right)C\xi_y\eta_y \\
&+ 4C^2\xi_y^2\eta_y^2 - 4A^2\xi_x^2\eta_x^2 - 4A\xi_x^2 B\eta_x\eta_y - 4A\xi_x^2 C\eta_y^2 \\
&- 4B\xi_x\xi_y A\eta_x^2 - 4B^2\xi_x\xi_y\eta_x\eta_y - 4B\xi_x\xi_y C\eta_y^2 \\
&- 4C\xi_y^2 A\eta_x^2 - 4C\xi_y^2 B\eta_x\eta_y - 4C^2\xi_y^2\eta_y^2.
\end{aligned}
$$

Collect terms to find

$$
\begin{aligned}
\Delta^* &= 4AB\xi_x^2\eta_x\eta_y + 4AB\xi_x\xi_y\eta_x^2 + 8AC\xi_x\xi_y\eta_x\eta_y \\
&+ B^2(\xi_x^2\eta_y^2 + 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2) \\
&+ 4BC\xi_x\xi_y\eta_y^2 + 4BC\eta_x\eta_y\xi_y^2 - 4AB\xi_x^2\eta_x\eta_y \\
&- 4AC\xi_x^2\eta_y^2 - 4AB\xi_x\xi_y\eta_x^2 - 4B^2\xi_x\xi_y\eta_x\eta_y \\
&- 4BC\xi_x\xi_y\eta_y^2 - 4AC\xi_y^2\eta_x^2 - 4BC\xi_y^2\eta_x\eta_y
\end{aligned}
$$

$$
\begin{aligned}
\Delta^* &= -4AC\left(\xi_x^2\eta_y^2 - 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2\right) \\
&+ B^2\left(\xi_x^2\eta_y^2 - 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2\right) \\
&= J^2\Delta,
\end{aligned}
$$

since $J = (\xi_x\eta_y - \xi_y\eta_x)$.

2.2 Canonical Forms

Problems

1. Find the characteristic equation, characteristic curves and obtain a canonical form for each

a. $x\, u_{xx} + u_{yy} = x^2$
b. $u_{xx} + u_{xy} - x u_{yy} = 0 \qquad (x \le 0, \text{ all } y)$
c. $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} + xy u_x + y^2 u_y = 0$
d. $u_{xx} + x u_{yy} = 0$
e. $u_{xx} + y^2 u_{yy} = y$
f. $\sin^2 x u_{xx} + \sin 2x u_{xy} + \cos^2 x u_{yy} = x$

2. Use Maple to plot the families of characteristic curves for each of the above.

1a. $xu_{xx} + u_{yy} = x^2$

$A = x$ $\quad B = 0$ $\quad C = 1$ $\quad \triangle = B^2 - 4AC = -4x$

If $x > 0$ then $\triangle < 0$ elliptic

$= 0$ $\quad = 0$ parabolic

< 0 $\quad > 0$ hyperbolic

characteristic equation

$$\frac{dy}{dx} = \frac{\pm\sqrt{-4x}}{2x} = \frac{\pm\sqrt{-x}}{x}$$

Suppose $x < 0$ (hyperbolic)

Let $z = -x$ (then $z > 0$)

then $dz = -dx$

and

$\frac{dy}{dz} = -\frac{dy}{dx} = -\frac{\pm\sqrt{z}}{-z} = \pm\frac{1}{\sqrt{z}}$

$dy = \pm\frac{dz}{z^{1/2}}$

$y = \pm 2\sqrt{z} + c$

$y \mp 2\sqrt{z} = c$

characteristic curves: $y \mp 2\sqrt{z} = c$
2 families as expected.

Transformation: $\xi = y - 2\sqrt{z}$

$\eta = y + 2\sqrt{z}$

$u_{xx} = u_{\xi\xi}\,\xi_x^2 + 2u_{\xi\eta}\,\xi_x\,\eta_x + u_{\eta\eta}\,\eta_x^2 + u_\xi\,\xi_{xx} + u_\eta\,\eta_{xx}$

$\xi_x = \xi_z\, z_x = -\xi_z$

$\eta_x = \eta_z\, z_x = -\eta_z$

$\xi_z = -2\left(\frac{1}{2}z^{-1/2}\right) = -\frac{1}{\sqrt{z}} \Rightarrow \xi_x = \frac{1}{\sqrt{z}}$

$$\eta_z = 2\left(\frac{1}{2}z^{-1/2}\right) = \frac{1}{\sqrt{z}} \Rightarrow \eta_x = -\frac{1}{\sqrt{z}}$$

$$\xi_y = 1$$

$$\eta_y = 1$$

$$\xi_{xx} = (\xi_x)_x = \left(\frac{1}{\sqrt{z}}\right)_x = \left(\frac{1}{\sqrt{z}}\right)_z z_x = -\left(-\frac{1}{2}z^{-3/2}\right) = \frac{1}{2z^{3/2}}$$

$$\eta_{xx} = (\eta_x)_x = \left(-\frac{1}{\sqrt{z}}\right)_x = \left(-\frac{1}{\sqrt{z}}\right)_z z_x = -\left(\frac{1}{2}z^{-3/2}\right) = \frac{-1}{2z^3/2}$$

$$\xi_{xy} = \xi_{yy} = \eta_{xy} = \eta_{yy} = 0$$

$$u_{xx} = \frac{1}{z}u_{\xi\xi} - \frac{2}{z}u_{\xi\eta} + \frac{1}{z}u_{\eta\eta} + \frac{1}{2z^{3/2}}u_\xi - \frac{1}{2z^{3/2}}u_\eta$$

$$u_{yy} = u_{\xi\xi}\underbrace{\xi_y^2}_{=1} + 2u_{\xi\eta}\,\xi_y\,\eta_y + u_{\eta\eta}\underbrace{\eta_y^2}_{=1} + u_\xi\underbrace{\xi_{yy}}_{=0} + u_\eta\underbrace{\eta_{yy}}_{=0}$$

$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Substitute in the equation

$$\underbrace{x}_{-z}\left\{\frac{1}{z}u_{\xi\xi} - \frac{2}{z}u_{\xi\eta} + \frac{1}{z}u_{\eta\eta} + \frac{1}{2z^{3/2}}u_\xi - \frac{1}{2z^{3/2}}u_\eta\right\} + u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} = \underbrace{x^2}_{(-z)^2}$$

$$-u_{\xi\xi} + 2u_{\xi\eta} - u_{\eta\eta} - \frac{1}{2z^{1/2}}u_\xi + \frac{1}{2z^{1/2}}u_\eta + u_{\xi\xi} + 2u_{\xi\eta} - u_{\eta\eta} = z^2$$

$$4u_{\xi\eta} - \frac{1}{2\sqrt{z}}u_\xi + \frac{1}{2\sqrt{z}}u_\eta = z^2$$

The last step is to get rid of z

$$\xi - \eta = -4\sqrt{z} \quad \text{(using the transformation)}$$

$$\sqrt{z} = \frac{\eta - \xi}{4} \Rightarrow 2\sqrt{z} = \frac{\eta - \xi}{2}\ ;\ z = \left(\frac{\eta - \xi}{4}\right)^2$$

$$\boxed{4u_{\xi\eta} - \frac{2}{\eta - \xi}u_\xi + \frac{2}{\eta - \xi}u_\eta = \left(\frac{\eta - \xi}{4}\right)^4}$$

For the elliptic case $\underline{x > 0}$

$$\frac{dy}{dx} = \frac{\pm i}{\sqrt{x}}$$

$$dy = \pm i \frac{dx}{\sqrt{x}}$$

$$y = \pm i\, 2\sqrt{x} + c$$

$$\xi = y - 2i\sqrt{x}$$

$$\eta = y + 2i\sqrt{x}$$

$$\alpha = \frac{1}{2}(\xi + \eta) = y$$

$$\beta = \frac{1}{2i}(\xi - \eta) = -2\sqrt{x}$$

$$u_{xx} = u_{\alpha\alpha}\, \alpha_x^2 + 2u_{\alpha\beta}\, \alpha_x\, \beta_x + u_{\beta\beta}\, \beta_x^2 + u_\alpha\, \alpha_{xx} + u_\beta\, \beta_{xx}$$

$$u_{yy} = u_{\alpha\alpha}\, \alpha_y^2 + 2u_{\alpha\beta}\, \alpha_y\, \beta_y + u_{\beta\beta}\, \beta_y^2 + u_\alpha\, \alpha_{yy} + u_\beta\, \beta_{yy}$$

$$\alpha_x = 0\, ;\ \alpha_y = 1\, ;\, \alpha_{xx} = \alpha_{yy} = 0$$

$$\beta_x = -2 \cdot \frac{1}{2} x^{-1/2} = -x^{-1/2}\, ;\, \beta_y = 0\, ;\, \beta_{xx} = \frac{1}{2} x^{-3/2}\, ;\, \beta_{yy} = 0$$

$$u_{xx} = u_{\beta\beta}(-x^{-1/2})^2 + u_\beta \left(\frac{1}{2}x^{-3/2}\right)$$

$$u_{yy} = u_{\alpha\alpha}$$

$$x\left[u_{\beta\beta} \cdot x^{-1} + \frac{1}{2} u_\beta\, x^{-3/2}\right] + u_{\alpha\alpha} = x^2$$

$$\boxed{u_{\beta\beta} + u_{\alpha\alpha} + \frac{1}{2} x^{-1/2}\, u_\beta = x^2}$$

Again, substitute for x:

$$-2\sqrt{x} = \beta$$

$$\Rightarrow \sqrt{x} = -\frac{1}{2}\beta$$

$$\Rightarrow x = \frac{1}{4}\beta^2$$

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{2}\frac{1}{-\frac{1}{2}\beta}\,u_\beta = \left(\frac{1}{4}\beta^2\right)^2$$

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{\beta}u_\beta + \frac{1}{16}\beta^4}$$

For the parabolic case $\underline{x = 0}$ the equation becomes:

$$0 \cdot u_{xx} + u_{yy} = 0$$

or $\boxed{u_{yy} = 0}$

which is already in a canonical form

This parabolic case can be solved. Integrate with respect to y holding x fixed (the constant of integration may depend on x)

$$u_y = f(x)$$

Integrate again:

$$\boxed{u(x,\,y) = y\,f(x) + g(x)}$$

1b. $u_{xx} + u_{xy} - x\,u_{yy} = 0$

A = 1 B = 1 C = -x

$$\triangle = 1 + 4x \qquad > 0 \qquad \text{if } x > -\tfrac{1}{4} \quad \underline{\text{hyperbolic}}$$

$$= 0 \qquad = -\tfrac{1}{4} \quad \underline{\text{parabolic}}$$

$$< 0 \qquad < -\tfrac{1}{4} \quad \underline{\text{elliptic}}$$

$$\frac{dy}{dx} = \frac{1 \pm \sqrt{1 + 4x}}{2}$$

Consider the hyperbolic case:

$$2dy = (1 \pm \sqrt{1 + 4x})\,dx$$

Integrate to get characteristics

$$2y = x \pm \frac{2}{3} \cdot \frac{1}{4}(1 + 4x)^{3/2} + c$$

$$2y - x \mp \frac{1}{6}(1 + 4x)^{3/2} = c$$

$$\xi = 2y - x - \frac{1}{6}(1 + 4x)^{3/2}$$

$$\eta = 2y - x + \frac{1}{6}(1 + 4x)^{3/2}$$

$$\xi_x = -1 - \frac{1}{6} \cdot \frac{3}{2} \cdot 4\,(1 + 4x)^{1/2} = -1 - \sqrt{1 + 4x}$$

$$\xi_{xx} = -\frac{1}{2}(1 + 4x)^{-1/2} \cdot 4 = -2\,(1 + 4x)^{-1/2}$$

$$\xi_y = 2 \qquad \xi_{yy} = 0 \qquad \xi xy = 0$$

$$\eta_x = -1 + \frac{1}{6} \cdot \frac{3}{2} \cdot 4\,(1 + 4x)^{1/2} = -1 + \sqrt{1 + 4x}$$

$$\eta_{xx} = +2\,(1 + 4x)^{-1/2}$$

$$\eta_y = 2 \qquad \eta_{xy} = 0 \qquad \eta_{yy} = 0$$

Now we can compute the new coefficients or compute each of the derivative in the equation. We chose the latter.

$u_{xx} \,=\, u_{\xi\xi}(-1 \,-\, \sqrt{1 \,+\, 4x})^2 \,+\, 2u_{\xi\eta}(-1 \,-\, \sqrt{1 \,+\, 4x})(-1 \,+\, \sqrt{1 \,+\, 4x})$

$+\, u_{\eta\eta}(-1 \,+\, \sqrt{1 \,+\, 4x})^2 \,+\, u_\xi\,[-2\,(1 \,+\, 4x)^{-1/2}] \,+\, u_\eta\,[2\,(1 \,+\, 4x)^{-1/2}]$

$=\, u_{\xi\xi}[1 \,+\, 2\,\sqrt{1 \,+\, 4x} \,+\, 1 \,+\, 4x] \,+\, 2u_{\xi\eta}(1 \,-\, (1 \,+\, 4x))$

$+\, u_{\eta\eta}[1 \,-\, 2\,\sqrt{1 \,+\, 4x} \,+\, 1 \,+\, 4x] \,-\, 2\,(1 \,+\, 4x)^{-1/2}\, u_\xi \,+\, 2(1 \,+\, 4x)^{-1/2}\, u_\eta$

$u_{xy} \,=\, 2u_{\xi\xi}\,(-1 \,-\, \sqrt{1 \,+\, 4x}) \,+\, u_{\xi\eta}\,[2\,(-1 \,-\, \sqrt{1 \,+\, 4x}) \,+\, 2\,(-1 \,+\, \sqrt{1 \,+\, 4x})]$

$+\, u_{\eta\eta}\, 2(-1 \,+\, \sqrt{1 \,+\, 4x})$

$u_{yy} \,=\, 4u_{\xi\xi} \,+\, 2u_{\xi\eta} \,\cdot\, 4 \,+\, u_{\eta\eta} \,\cdot\, 4$

$\Rightarrow\, u_{xx} \,+\, u_{xy} \,-\, x\,u_{yy} \,=$

$u_{\xi\xi}\,[2 \,+\, 4x \,+\, 2\,\sqrt{1 \,+\, 4x}] \,+\, 2u_{\xi\eta}\,(-\,4x) \,+\, u_{\eta\eta}\,(2 \,+\, 4x \,-\, 2\,\sqrt{1 \,+\, 4x})$

$-\,2\,(1 \,+\, 4x)^{-1/2}\, u_\xi \,+\, 2\,(1 \,+\, 4x)^{-1/2}\, u_\eta$

$+\,2u_{\xi\xi}\,(-1 \,-\, \sqrt{1 \,+\, 4x}) \,+\, u_{\xi\eta}\,(-4) \,+\, 2u_{\eta\eta}\,(-1 \,+\, \sqrt{1 \,+\, 4x}) \,-$

$4x\,(u_{\xi\xi} \,+\, 2u_{\xi\eta} \,+\, u_{\eta\eta}) \,=$

$$\begin{aligned}&(2 + 4x + 2\sqrt{1+4x} - 2 - 2\sqrt{1+4x} - 4x)\,u_{\xi\xi} \,+\, (-8x - 4 - 8x)u_{\xi\eta}\\ &+ (2 + 4x - 2\sqrt{1+4x} - 2 - 2\sqrt{1+4x} - 4x)\,u_{\eta\eta} -\, 2(1 \,+\, 4x)^{-1/2}\,(u_\xi \,-\, u_\eta) \,=\, 0\end{aligned}$$

$-4\,(1 + 4x)u_{\xi\eta} \,-\, 2(1 \,+\, 4x)^{-1/2}\,(u_\xi \,-\, u_\eta) \,=\, 0$

$u_{\xi\eta} \,+\, \dfrac{1}{2}(1 \,+\, 4x)^{-3/2}\,(u_\xi \,-\, u_\eta) \,=\, 0$

Now find $(1 \,+\, 4x)^{-3/2}$ in terms of ξ, η and substitute

$\xi \,-\, \eta \,=\, -\dfrac{1}{3}\,(1 \,+\, 4x)^{3/2}$

$3(\eta \,-\, \xi) \,=\, (1 \,+\, 4x)^{3/2}$

$(1 \,+\, 4x)^{-3/2} \,=\, [3(\eta \,-\, \xi)]^{-1}$

$u_{\xi\eta} \,=\, -\dfrac{1}{2[3(\eta \,-\, \xi)]}\,(u_\xi \,-\, u_\eta)$

$$u_{\xi\eta} = \frac{1}{6(\eta - \xi)}(u_\eta - u_\xi)$$

The parabolic case is easier, the only characteristic is

$$y = \frac{1}{2}x + K$$

and so the transformation is

$$\xi = y - \frac{1}{2}x$$
$$\eta = x$$

The last equation is an arbitrary function and one should check the Jacobian. The details are left to the reader. One can easily show that

$$A^* = B^* = 0$$

Also

$$C^* = 1$$

and the rest of the coefficients are zero. Therefore the equation is

$$u_{\eta\eta} = 0$$

In the elliptic case, one can use the transformation $z = -(1+4x)$ so that the characteristic equation becomes

$$\frac{dy}{dx} = \frac{1 \pm \sqrt{z}}{2}$$

or if we eliminate the x dependence

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = -\frac{1}{4}\frac{1 \pm \sqrt{z}}{2}$$

Now integrate, and take the real and imaginary part to be the functions ξ and η. The rest is left for the reader.

1c. $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + xy\, u_x + y^2 u_y = 0$

$A = x^2 \qquad B = 2xy \qquad C = y^2$

$\triangle = 4x^2 y^2 - 4x^2 y^2 = 0$ <u>parabolic</u>

$$\frac{dy}{dx} = \frac{2xy}{2x^2} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\xi = \ell n\, y - \ell n\, x \quad \Rightarrow \quad \xi = \ell n \left(\frac{y}{x}\right) \Rightarrow e^{\xi} = \frac{y}{x}$$

$\eta = x$ arbitrarily chosen since this is parabolic

$$\xi_x = \frac{-1}{x} \qquad \xi_y = \frac{1}{y} \qquad \xi_{xx} = \frac{1}{x^2} \qquad \xi_{xy} = 0 \qquad \xi_{yy} = -\frac{1}{y^2}$$

$$\eta_x = 1 \quad \eta_y = \eta_{xx} = \eta_{xy} = \eta_{yy} = 0$$

$$u_{xx} = \frac{1}{x^2} u_{\xi\xi} + 2u_{\xi\eta}(-\frac{1}{x}) + u_{\eta\eta} + \frac{1}{x^2} u_{\xi}$$

$$u_{xy} = -\frac{1}{xy} u_{\xi\xi} + u_{\xi\eta}\frac{1}{y}$$

$$u_{yy} = \frac{1}{y^2} u_{\xi\xi} - \frac{1}{y^2} u_{\xi}$$

$$u_{\xi\xi} - 2xu_{\xi\eta} + x^2 u_{\eta\eta} + u_{\xi} - 2u_{\xi\xi} + 2xu_{\xi\eta} + u_{\xi\xi} - u_{\xi} +$$
$$xy(-\frac{1}{x} u_{\xi} + u_{\eta}) + y^2(\frac{1}{y} u_{\xi}) = 0$$

$$x^2 u_{\eta\eta} + xyu_{\eta} = 0$$

$\boxed{u_{\eta\eta} = -e^{\xi} u_{\eta}}$ $\qquad y = e^{\xi} x$ therefore $y/x = e^{\xi}$

This equation can be solved.

1d. $u_{xx} + x\,u_{yy} = 0$

A = 1 B = 0 C = x

$\triangle = -4x$	> 0	if $x < 0$	<u>hyperbolic</u>
	$= 0$	$x = 0$	<u>parabolic</u>
	< 0	$x > 0$	<u>elliptic</u>

<u>Parabolic</u> $x = 0 \quad \Rightarrow \quad u_{xx} = 0$ already in canonical form

<u>Hyperbolic</u> $x < 0$ Let $\zeta = -x$

$$\triangle = 4\zeta > 0$$

$$\frac{dy}{dx} = \pm \frac{2\sqrt{\zeta}}{2} = \pm\sqrt{\zeta} \qquad \text{Note: } dx = -d\zeta$$

$$dy = \pm\sqrt{\zeta}\,(-d\zeta)$$

$$y \pm \frac{2}{3}\zeta^{3/2} = c$$

$$\xi = y + \frac{2}{3}\zeta^{3/2}$$

$$\eta = y - \frac{2}{3}\zeta^{3/2}$$

Continue as in example in class (See 1a)

1e. $u_{xx} + y^2 u_{yy} = y$

$A = 1 \qquad B = 0 \qquad C = y^2$

$\triangle = -4y^2 < 0$ <u>elliptic</u> if $y \neq 0$

For $y = 0$ the equation is <u>parabolic</u> and it is in canonical form $u_{xx} = 0$

$$\frac{dy}{dx} = \frac{\pm\sqrt{-4y^2}}{2} = \pm iy$$

$$\frac{dy}{y} = \pm i dx$$

$$\ln y = \pm ix + c$$

$$\xi = \ln y + ix$$

$$\eta = \ln y - ix$$

$$\alpha = \ln y \qquad \alpha_x = 0 \qquad \alpha_y = \frac{1}{y}$$

$$\beta = x \qquad \beta_x = 1 \qquad \beta_y = 0$$

$$u_x = u_\beta \beta_x + u_\alpha \alpha_x = u_\beta$$

$$u_y = u_\alpha \frac{1}{y} + u_\beta \beta_y = \frac{1}{y} u_\alpha$$

$$u_{xx} = (u_\beta)_x = u_{\beta\beta}$$

$$u_{yy} = \left(\frac{1}{y}\right)_y u_\alpha + \frac{1}{y}(u_\alpha)_y = -\frac{1}{y^2} u_\alpha + \frac{1}{y^2} u_{\alpha\alpha}$$

$$\Rightarrow \quad u_{\beta\beta} + y^2\left(-\frac{1}{y^2} u_\alpha + \frac{1}{y^2} u_{\alpha\alpha}\right) = y$$

$$u_{\alpha\alpha} + u_{\beta\beta} - u_\alpha = y$$

But $y = e^\alpha$

$$\boxed{\Rightarrow \quad u_{\alpha\alpha} + u_{\beta\beta} - u_\alpha = e^\alpha}$$

1f. $\sin^2 x\, u_{xx} + \sin 2x\, u_{xy} + \cos^2 x\, u_{yy} = x$

$A = \sin^2 x \qquad B = \sin 2x = 2\sin x \cos x \qquad C = \cos^2 x$

$\triangle = 0$ <u>parabolic</u>

$$\frac{dy}{dx} = \frac{2\sin x \cos x}{2\sin^2 x} = \cot x$$

$y = \ln \sin x + c$

$\xi = y - \ln \sin x \qquad \xi_x = -\cot x \qquad \xi_y = 1$

$\eta = y \qquad \eta_x = 0 \qquad \eta_y = 1$

$u_x = -\cot x\, u_\xi + u_\eta\, \eta_x = -\cot x\, u_\xi$

$u_y = u_\xi + u_\eta$

$$u_{xx} = (-\cot x\, u_\xi)_x = \frac{1}{\sin^2 x} u_\xi + \cot^2 x\, u_{\xi\xi}$$

$u_{xy} = -\cot x\, (u_\xi)_y = -\cot x\, (u_{\xi\xi} + u_{\xi\eta})$

$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$$L.\ H.\ S. = u_\xi + \sin^2 x \frac{\cos^2 x}{\sin^2 x} u_{\xi\xi} + 2\sin x \cos x\,(-\cot x)(u_{\xi\xi} + u_{\xi\eta})$$
$$+ \cos^2 x\,(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$

$L.\ H.\ S. = \cos^2 x\, u_{\eta\eta} + u_\xi$

Therefore the equation becomes:

$\cos^2 x\, u_{\eta\eta} + u_\xi = x$

$\ln \sin x = y - \xi = \eta - \xi$

$\sin x = e^{\eta - \xi} \quad \Rightarrow \quad \cos^2 x = 1 - \sin^2 x = 1 - e^{2(\eta-\xi)}$

$x = \arcsin e^{\eta - \xi}$

$$\boxed{[1 - e^{2(\eta-\xi)}]\, u_{\eta\eta} + u_\xi = \arcsin e^{\eta-\xi}}$$

2a. $y \pm 2\sqrt{z} = c \qquad z > 0$

eq: $y + 2 * sqrt(z) = c;$ $\longleftarrow$ maple command to give the equation

char:=solve (eq, y); $\longleftarrow$ maple command to solve for y

chars:=seq (char, c= -5..5); $\longleftarrow$ maple command to create several characteristic curves for a variety of c's.

plot ($\{$chars$\}$, $z = 0..10, y = -5..5$); $\longleftarrow$ maple command to plot all those curves

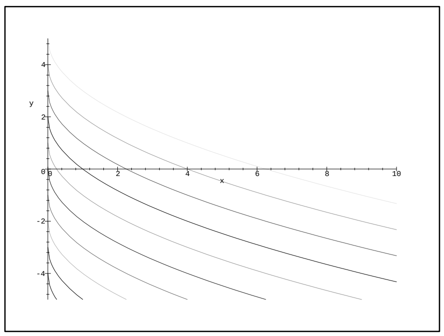

Figure 1: Maple plot of characteristics for 2.2 2a

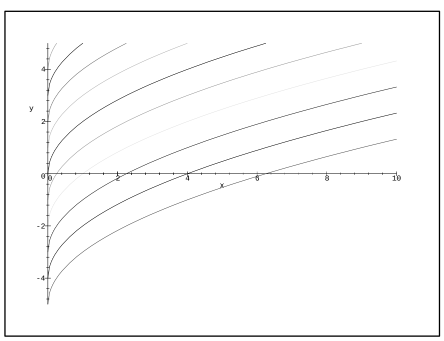

Figure 2: Maple plot of characteristics for 2.2 2a

2b. $y = \frac{1}{2}x \pm \frac{1}{12}(1 + 4x)^{3/2} + c$

$$1 + 4\,x \geq 0$$

$$4\,x \geq -1$$

$$x \geq -.25$$

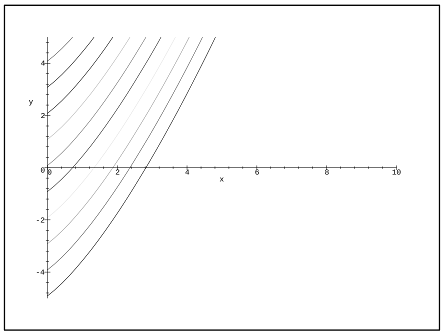

Figure 3: Maple plot of characteristics for 2.2 2b

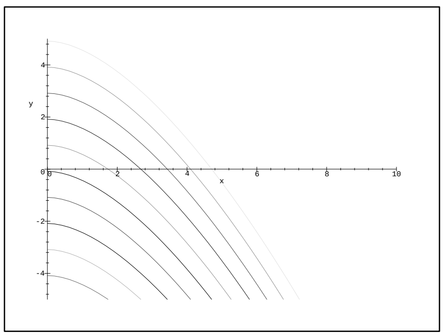

Figure 4: Maple plot of characteristics for 2.2 2b

2c. $\ln \dfrac{y}{x} = c$ parabolic

$$\ln y = xe^c = kx$$

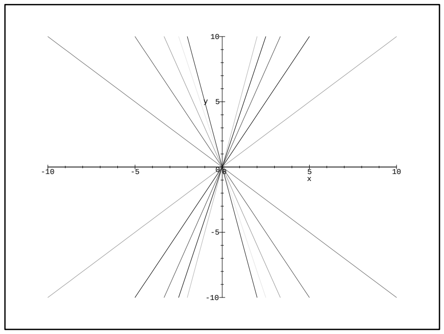

Figure 5: Maple plot of characteristics for 2.2 2c

2d. $\quad y \pm \dfrac{2}{3} z^{3/2} = c$

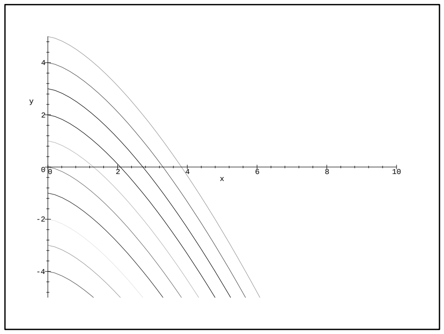

Figure 6: Maple plot of characteristics for 2.2 2d

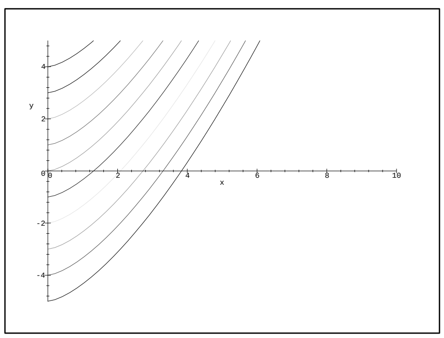

Figure 7: Maple plot of characteristics for 2.2 2d

2e. elliptic. no real characteristic

2f. $y = \ln \sin x + c$

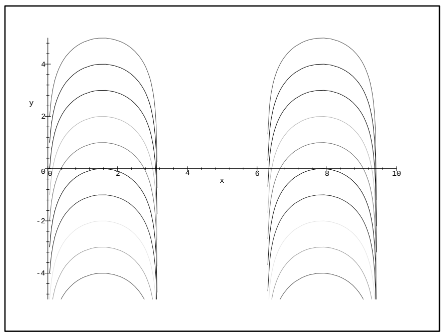

Figure 8: Maple plot of characteristics for 2.2 2f

2.3 Equations with Constant Coefficients

Problems

1. Find the characteristic equation, characteristic curves and obtain a canonical form for

   a. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$
   b. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$
   c. $3u_{xx} + 10u_{xy} + 3u_{yy} = x + 1$
   d. $u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$
   e. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$
   f. $u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$

2. Use Maple to plot the families of characteristic curves for each of the above.

1a. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$

A $=4$

B $=5$

C $=1$

$\triangle = 5^2 - 4 \cdot 4 \cdot 1 = 25 - 16 = 9 > 0$ $\quad$ <u>hyperbolic</u>

$$\frac{dy}{dx} = \frac{5 \pm \sqrt{9}}{2 \cdot 4} = \frac{5 \pm 3}{8} \begin{array}{l} \nearrow 1/4 \\ \searrow 1 \end{array}$$

$$dy = dx \qquad dy = \frac{1}{4}\,dx$$

$$y = x + c \qquad y = \frac{1}{4}\,x + c$$

$$\xi = y - x \qquad \eta = y - \frac{1}{4}x$$

$$\xi_x = -1 \qquad \xi_y = 1 \qquad \eta_x = -\frac{1}{4} \qquad \eta_y = 1$$

$$\xi_{xx} = 0 \qquad \xi_{yy} = 0 \qquad \xi_{xy} = 0 \qquad \eta_{xx} = 0 \qquad \eta_{yy} = 0 \qquad \eta_{xy} = 0$$

$$u_{xx} = u_{\xi\xi}\,(-1)^2 + 2u_{\xi\eta}\,(-1)\left(-\frac{1}{4}\right) + u_{\eta\eta}\left(-\frac{1}{4}\right)^2 + u_\xi \cdot 0 + u_\eta \cdot 0$$

$$u_{yy} = u_{\xi\xi} \cdot 1^2 + 2u_{\xi\eta} \cdot 1 \cdot 1 \quad + u_{\eta\eta} \cdot 1^2 + u_\xi \cdot 0 + u_\eta \cdot 0$$

$$u_{xy} = u_{\xi\xi}\,(-1) \cdot 1 + u_{\xi\eta}\left(-1 \cdot 1 + 1 \cdot \left(-\frac{1}{4}\right)\right) + u_{\eta\eta}\left(-\frac{1}{4}\right) \cdot 1 + u_\xi \cdot 0 \cdot + u_\eta \cdot 0$$

$$u_x = u_\xi\,(-1) + u_\eta\left(-\frac{1}{4}\right)$$

$$u_y = u_\xi \cdot 1 + u_\eta \cdot 1$$

$$4\,u_{\xi\xi} + 2u_{\xi\eta} + \frac{1}{4}u_{\eta\eta} - 5\,u_{\xi\xi} - \frac{25}{4}\,u_{\xi\eta} - \frac{5}{4}\,u_{\eta\eta} + u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$-\,u_\xi - \frac{1}{4}\,u_\eta + u_\xi + u_\eta \qquad = 2$$

All $u_{\xi\xi}$, $u_{\eta\eta}$ and u_ξ terms cancel out

$$-\frac{9}{4}\,u_{\xi\eta} + \frac{3}{4}\,u_\eta = 2$$

$$\boxed{u_{\xi\,\eta} \;=\; \tfrac{1}{3}\,u_\eta \;-\; \tfrac{8}{9}}$$

This equation can be solved as follows:

Let $\nu \;=\; u_\eta$ then $u_{\xi\,\eta} \;=\; \nu_\xi$

$\nu_\xi \;=\; \frac{1}{3}\,\nu \;-\; \frac{8}{9}$

This is Linear 1^{st} order ODE

$$\nu' \;-\; \frac{1}{3}\nu \;=\; -\,\frac{8}{9}$$

Integrating factor is $e^{-\frac{1}{3}\,\xi}$

$$(\nu e^{-\frac{1}{3}\,\xi})' \;=\; -\,\frac{8}{9}\,e^{-\frac{1}{3}\,\xi}$$

$$\nu e^{-\frac{1}{3}\,\xi} \;=\; -\,\frac{8}{9}\int\,e^{-\frac{1}{3}\,\xi}\,d_\xi \;=\; \frac{8}{3}\,e^{-\frac{1}{3}\,\xi}\;+\;C\,(\eta)$$

$$\boxed{\nu \;=\; \tfrac{8}{3}\;+\;C\,(\eta)\,e^{\frac{1}{3}\,\xi}}$$

To find u we integrate with respect to η

$$u_\eta \;=\; \frac{8}{3}\;+\;C(\eta)\,e^{\frac{1}{3}\,\xi}$$

$$u \;=\; \frac{8}{3}\,\eta\;+\;e^{\frac{1}{3}\,\xi}\qquad\underbrace{c_1(\eta)}_{\text{integral of } C(\eta)}\qquad+\,K(\xi)$$

To check the solution, we differentiate it and substitute in the canonical form:

$$u_\xi \;=\; 0\;+\;\frac{1}{3}\,e^{\frac{1}{3}\,\xi}\,c_1\,(\eta)\;+\;K'\,(\xi)$$

$$u_{\xi\,\eta} \;=\; \frac{1}{3}\,e^{\frac{1}{3}\,\xi}\,c_1'\,(\eta)$$

$$u_\eta \;=\; \frac{8}{3}\;+\;e^{\frac{1}{3}\,\xi}\,c_1'\,(\eta)$$

$$\Rightarrow \quad \frac{1}{3}\,u_\eta \,=\, \frac{8}{9} \,+\, \frac{1}{3}\,e^{\frac{1}{3}\,\xi}\,c_1'\,(\eta)$$

Substitute in the PDE in canonical form

$$\frac{1}{3}\,e^{\frac{1}{3}\,\xi}\,c_1'\,(\xi) \,=\, \frac{8}{9} \,+\, \frac{1}{3}\,e^{\frac{1}{3}\,\xi}\,c_1'\,(\eta) \,-\, \frac{8}{9}$$

Identity

$$\boxed{\text{In terms of original variables} \qquad u(x,\,y) \,=\, \tfrac{8}{3}\,(y \,-\, \tfrac{1}{4}x) \,+\, e^{\frac{1}{3}\,(y-x)}\,c_1\,(y \,-\, \tfrac{1}{4}x) \,+\, K\,(y-x)}$$

1b. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$

A = 1 B = 1 C = 1 $\triangle = 1 - 4 = -3 < 0$ <u>elliptic</u>

$$\frac{dy}{dx} = \frac{1 \pm \sqrt{-3}}{2}$$

$$2dy = (1 \pm \sqrt{3}i)\, dx$$

$$\xi = 2y - (1 + \sqrt{3}i)x \qquad \eta = 2y - (1 - \sqrt{3}i)x$$

$$\alpha = \frac{1}{2}(\xi + \eta) = 2y - x$$

$$\beta = \frac{1}{2i}(\xi - \eta) = -\sqrt{3}x$$

$$\alpha_x = -1 \qquad \alpha_y = 2 \qquad \alpha_{xx} = 0 \qquad \alpha_{xy} = 0 \qquad \alpha_{yy} = 0$$

$$\beta_x = -\sqrt{3} \qquad \beta_y = 0 \qquad \beta_{xx} = 0 \qquad \beta_{xy} = 0 \qquad \beta_{yy} = 0$$

$$u_{\alpha\alpha} + 2u_{\alpha\beta}(-1)(-\sqrt{3}) + u_{\beta\beta} \cdot 3 + \underbrace{u_{\alpha\alpha}(-2) + u_{\alpha\beta}(-2\sqrt{3})}_{u_{xy}} + 4u_{\alpha\alpha} - u_\alpha - \sqrt{3}u_\beta = 0$$

$$3u_{\alpha\alpha} + 3u_{\beta\beta} - u_\alpha - \sqrt{3}\, u_\beta = 0$$

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{3}u_\alpha + \frac{\sqrt{3}}{3}\, u_\beta}$$

1c. $3u_{xx} + 10u_{xu} + 3u_{yy} = x + 1$

$\mathrm{A} = \mathrm{C} = 3 \qquad \mathrm{B} = 10 \qquad \triangle = 100 - 36 = 64 > 0 \qquad \underline{\text{hyperbolic}}$

$$\frac{dy}{dx} = \frac{10 \pm 8}{6} \begin{matrix} \nearrow 3 \\ \searrow 1/3 \end{matrix}$$

$$\xi = y - 3x \qquad \eta = y - \frac{1}{3}x$$

$$\xi_x = -3 \qquad \xi_y = 1 \qquad \xi_{xx} = 0 \qquad \xi_{xy} = 0 \qquad \xi_{yy} = 0$$

$$\eta_x = -\frac{1}{3} \qquad \eta_y = 1 \qquad \eta_{xx} = 0 \qquad \eta_{xy} = 0 \qquad \eta_{yy} = 0$$

$$3\left(u_{\xi\xi}(-3)^2 + 2u_{\xi\eta}(-3)\left(-\frac{1}{3}\right) + u_{\eta\eta}\left(-\frac{1}{3}\right)^2\right)$$

$$+10\left(u_{\xi\xi}(-3) + u_{\xi\eta}\left(-3 - \frac{1}{3}\right) + u_{\eta\eta}\left(-\frac{1}{3}\right)\right)$$

$$+3\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right) = x + 1$$

$$-\tfrac{64}{3}u_{\xi\eta} = x + 1$$

$$\left.\begin{aligned} \xi &= y - 3x \\ \eta &= y - \frac{1}{3}x \end{aligned}\right\} -$$

$$\xi - \eta = -\frac{8}{3}x$$

$$x = \frac{3}{8}(\eta - \xi)$$

$$-\frac{64}{3}u_{\xi\eta} = \frac{3}{8}(\eta - \xi) + 1$$

$$\boxed{u_{\xi\eta} = -\tfrac{9}{512}(\eta - \xi) - \tfrac{3}{64}}$$

To Find the general solution !

$$u_{\xi\,\eta} \;=\; -\,\frac{9}{512}\,(\eta\,-\,\xi)\;-\;\frac{3}{64}$$

$$u_{\xi} \;=\; -\,\frac{9}{512}\,(\frac{1}{2}\eta^2\,-\,\eta\,\xi)\;-\;\frac{3}{64}\eta\,+\,f(\xi)$$

$$u \;=\; -\,\frac{9}{512}\,(\frac{1}{2}\eta^2\,\xi\,-\,\frac{1}{2}\,\xi^2\,\eta)\;-\;\frac{3}{64}\eta\,\xi\,+\,F\,(\xi)\,+\,G\,(\eta)$$

$$\;=\;\frac{9}{512}\,\cdot\,\frac{1}{2}\eta\,\xi(\xi\,-\,\eta)\;-\;\frac{3}{64}\,\eta\,\xi\,+\,F\,(\xi)\,+\,G\,(\eta)$$

$$u(x,\,y)\;=\;\frac{9}{1024}\left(y\,-\,\frac{1}{3}x\right)\,(y\,-\,3x)\underbrace{\left(\frac{1}{3}x\,-\,3x\right)}_{-\frac{8}{3}x}-\frac{3}{64}\left(y\,-\,\frac{1}{3}x\right)\,(y\,-\,3x)$$

$$+\,F\,(y\,-\,3x)\,+\,G\,(y-\tfrac{1}{3}x)$$

$$=\;\frac{9}{1024}\,\cdot\,\frac{-8}{3}x\left(y\,-\,\frac{1}{3}x\right)\,(y\,-\,3x)\;-\;\frac{3}{64}\left(y\,-\,\frac{1}{3}x\right)\,(y\,-\,3x)\,+\,F\,(y\,-\,3x)$$
$$+\,G\,(y-\tfrac{1}{3}x)$$

$$\boxed{u(x\,,\,y)\;=\;\left(-\frac{3}{128}x\,-\,\frac{3}{64}\right)(y\,-\,\frac{1}{3}x)\,(y\,-\,3x)+\,F\,(y\,-\,3x)\,+\,G\left(y-\frac{1}{3}x\right)}$$

check !

$$u_x\;=\;-\frac{3}{128}\,(y\,-\,\frac{1}{3}x)\,(y\,-\,3x)+\,(-\frac{3}{128}x\,-\,\frac{3}{64})\,(-\frac{1}{3})\,(y\,-\,3x)$$

$$+\left(-\frac{3}{128}x\,-\,\frac{3}{64}\right)\left(y\,-\,\frac{1}{3}x\right)\,(-3)\,-\,3F'\,(y\,-\,3x)\,-\frac{1}{3}\,G'\left(y-\frac{1}{3}x\right)$$

$$u_y\;=\;\left(-\frac{3}{128}x\,-\,\frac{3}{64}\right)\,(y-3x)+\left(-\frac{3}{128}x\,-\,\frac{3}{64}\right)\left(y-\frac{1}{3}x\right)+F'\,(y-3x)+G'\left(y-\frac{1}{3}x\right)$$

$$u_{xx}\;=\;-\frac{3}{128}\left(-\frac{1}{3}\right)\,(y\,-\,3x)\,+\,\frac{9}{128}\,(y-\frac{1}{3}x)\,+\,(-\frac{3}{128}x\,-\,\frac{3}{64})\,-\,\frac{1}{3}\,(-\frac{3}{128})\,(y\,-\,3x)$$

$$-3\left(-\frac{3}{128}\right)\left(y-\frac{1}{3}x\right)\,-3\,(-\frac{1}{3})\,(-\frac{3}{128}x\,-\,\frac{3}{64})\,+\,9F''\,+\,\frac{1}{9}G''$$

$$u_{xx}\;=\;\frac{1}{64}\,(y\,-\,3x)+\,\frac{9}{64}\,(y-\frac{1}{3}x)\,+2\,(-\frac{3}{128}x\,-\,\frac{3}{64})\,+9F''\,(y\,-\,3x)\,+\,\frac{1}{9}G''(y-\frac{1}{3}x)$$

$$u_{xy} = -\frac{3}{128}(y - 3x) - 3(-\frac{3}{128}x - \frac{3}{64}) - \frac{3}{128}(y - \frac{1}{3}x) - \frac{1}{3}(-\frac{3}{128}x - \frac{3}{64})$$

$$-3F''(y - 3x) - \frac{1}{3}G''(y - \frac{1}{3}x)$$

$$u_{yy} = -\frac{3}{128}x - \frac{3}{64} - \frac{3}{128}x - \frac{3}{64} + F''(y - 3x) + G''\left(y - \frac{1}{3}x\right)$$

$$3u_{xx} + 10u_{xy} + 3u_{yy} = \frac{3}{64}(y - 3x) + \frac{27}{64}(y - \frac{1}{3}x) + 6\left(-\frac{3}{128}x - \frac{3}{64}\right) + 27F'' + \frac{1}{3}G''$$

$$-\frac{30}{128}(y - 3x) - \frac{15}{64}(y - \frac{1}{3}x) - \frac{100}{3}\left(-\frac{3}{128}x - \frac{3}{64}\right) - 30F'' - \frac{10}{3}G''$$

$$+6\left(-\frac{3}{128}x - \frac{3}{64}\right) + 3F'' + 3G''$$

$$= -\frac{12}{64}(y - 3x) + \frac{12}{64}(y - \frac{1}{3}x) - \frac{64}{3}(-\frac{3}{128}x - \frac{3}{64})$$

$$= \frac{9}{16}x - \frac{1}{16}x + \frac{1}{2}x + 1 \qquad = x + 1$$

checks

1d. $u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$

A = 1 B = 2 C = 3 $\triangle = 4 - 12 = -8 < 0$ elliptic

$$\frac{dy}{dx} = \frac{2 \pm \sqrt{-8}}{2} = 1 \pm i\sqrt{2}$$

$$y = (1 \pm i\sqrt{2})x + C$$

$$\xi = y - (1 + i\sqrt{2})x$$

$$\eta = y - (1 - i\sqrt{2})x$$

$$\alpha = y - x$$

$$\beta = -\sqrt{2}x \qquad \Rightarrow x = -\frac{\beta}{\sqrt{2}}$$

$$\alpha_x = -1 \qquad \alpha_y = 1 \qquad \alpha_{xx} = \alpha_{xy} = \alpha_{yy} = 0$$

$$\beta_x = -\sqrt{2} \qquad \beta_y = 0 \qquad \beta_{xx} = \beta_{xy} = \beta_{yy} = 0$$

$$u_{\alpha\alpha}(-1)^2 + 2u_{\alpha\beta} \cdot \sqrt{2} + u_{\beta\beta} \cdot 2$$

$$+2\left(-u_{\alpha\alpha} + u_{\alpha\beta}(-\sqrt{2})\right) + 3\,u_{\alpha\alpha} + 4\left(-u_\alpha - \sqrt{2}\,u_\beta\right) + 5u_\alpha + u = e^x$$

$$2u_{\alpha\alpha} + 2u_{\beta\beta} + u_\alpha - 4\sqrt{2}\,u_\beta + u = e^x$$

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = -\tfrac{1}{2}u_\alpha + 2\sqrt{2}\,u_\beta - \tfrac{1}{2}u + \tfrac{1}{2}e^{-\beta/\sqrt{2}}}$$

1e. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$

A = C = 2 B = -4 $\triangle = 16 - 16 = 0$ parabolic

$$\frac{dy}{dx} = \frac{-4 \pm 0}{4} = -1$$

$dy = -dx$

$$\begin{cases} \xi = y + x & \xi_x = 1 & \xi_y = 1 & \xi_{xx} = \xi_{xy} = \xi_{yy} = 0 \\ \eta = x & \eta_x = 1 & \eta_y = 0 & \eta_{xx} = \eta_{xy} = \eta_{yy} = 0 \end{cases}$$

$2\,(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) - 4\,(u_{\xi\xi} + u_{\xi\eta}) + 2\,u_{\xi\xi} + 3u = 0$

$2u_{\eta\eta} + 3u = 0$

$\boxed{u_{\eta\eta} = -\frac{3}{2}u}$

1f. $u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$

A = 1 B = 5 C = 4 $\triangle = 25 - 16 = 9 > 0$ <u>hyperbolic</u>

$$\frac{dy}{dx} = \frac{5 \pm 3}{2} \begin{matrix} \nearrow 4 \\ \searrow 1 \end{matrix}$$

$$\begin{cases} \xi = y - 4x & \xi_x = -4 \quad \xi_y = 1 \quad \xi_{xx} = \xi_{xy} = \xi_{yy} = 0 \\ \eta = y - x & \eta_x = -1 \quad \eta_y = 1 \quad \eta_{xx} = \eta_{xy} = \eta_{yy} = 0 \end{cases}$$

$$16u_{\xi\xi} + 2u_{\xi\eta} \cdot 4 + u_{\eta\eta} + 5\left(-4u_{\xi\xi} + u_{\xi\eta}(-4-1) + u_{\eta\eta}(-1)\right)$$

$$+4\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right) + 7\left(u_\xi + u_\eta\right) = \sin x$$

$$-9\,u_{\xi\eta} + 7(u_\xi + u_\eta) = \sin x$$

$$u_{\xi\eta} = \frac{7}{9}\left(u_\xi + u_\eta\right) - \frac{1}{9}\sin x$$

$$\xi - \eta = -3x$$

$$x = \frac{\eta - \xi}{3}$$

$$\boxed{u_{\xi\eta} = \tfrac{7}{9}\left(u_\xi + u_\eta\right) - \tfrac{1}{9}\sin\tfrac{\eta-\xi}{3}}$$

2a. $y = x + c$

$y = \frac{1}{4}x + c$

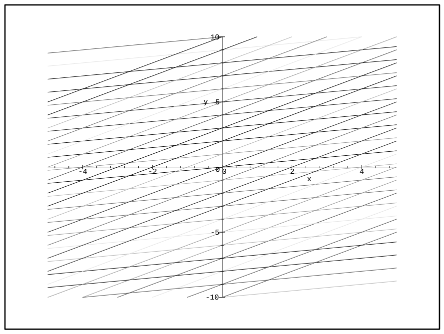

Figure 9: Maple plot of characteristics for 2.3 2a

2b. elliptic . no real characteristics

2c. $y = 3x + c$

$y = \frac{1}{3}x + c$

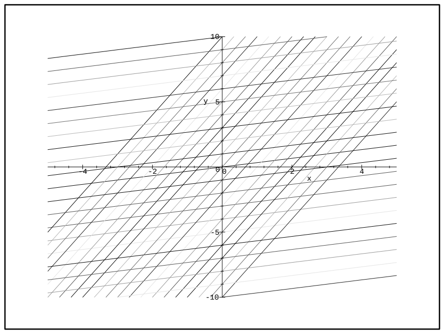

Figure 10: Maple plot of characteristics for 2.3 2c

2d. elliptic . no real characteristics

2e. $y = x + c$ see 2a

2f. $y = 4x + c$

$y = x + c \qquad \rightarrow \qquad$ (see 2a)

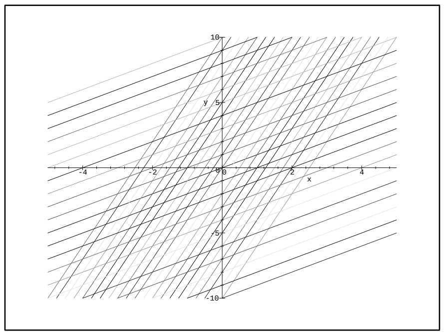

Figure 11: Maple plot of characteristics for 2.3 2f

2.4 General Solution

Problems

1. Determine the general solution of

   a. $u_{xx} - \frac{1}{c^2} u_{yy} = 0 \qquad c = \text{constant}$
   b. $u_{xx} - 3u_{xy} + 2u_{yy} = 0$
   c. $u_{xx} + u_{xy} = 0$
   d. $u_{xx} + 10u_{xy} + 9u_{yy} = y$

2. Transform the following equations to

$$U_{\xi\eta} = cU$$

by introducing the new variables

$$U = ue^{-(\alpha\xi + \beta\eta)}$$

where α, β to be determined

   a. $u_{xx} - u_{yy} + 3u_x - 2u_y + u = 0$
   b. $3u_{xx} + 7u_{xy} + 2u_{yy} + u_y + u = 0$

(Hint: First obtain a canonical form)

3. Show that

$$u_{xx} = au_t + bu_x - \frac{b^2}{4} u + d$$

is parabolic for a, b, d constants. Show that the substitution

$$u(x,t) = v(x,t)e^{\frac{b}{2}x}$$

transforms the equation to

$$v_{xx} = av_t + de^{-\frac{b}{2}x}$$

1a. $u_{xx} - \frac{1}{c^2} u_{yy} = 0$

$A = 1 \qquad B = 0 \qquad C = -\frac{1}{c^2} \qquad \triangle = \frac{4}{c^2} > 0 \qquad \underline{\text{hyperbolic}}$

$\frac{dy}{dx} = \frac{\pm \frac{2}{c}}{2} = \pm \frac{1}{c}$

$y = \pm \frac{1}{c} x + K$

$\xi = y + \frac{1}{c} x$

$\eta = y - \frac{1}{c} x$

Canonical form:

$u_{\xi \eta} = 0$

The solution is:

$u = f(\xi) + g(\eta)$

Substitute for ξ and η to get the solution in the original domain:

$u(x, y) = f(y + \frac{1}{c} x) + g(y - \frac{1}{c} x)$

1b. $u_{xx} - 3u_{xy} + 2u_{yy} = 0$

$A = 1 \qquad B = -3 \qquad C = 2 \qquad \triangle = 9 - 8 = 1 \qquad \underline{\text{hyperbolic}}$

$$\frac{dy}{dx} = \frac{-3 \pm 1}{2} \begin{array}{l} \nearrow -2 \\ \searrow -1 \end{array}$$

$y = -2x + K_1$

$y = -x + K_2$

$\xi = y + 2x \qquad \xi_x = 2 \qquad \xi_y = 1$

$\eta = y + x \qquad \eta_x = 1 \qquad \eta_y = 1$

$u_x = 2u_\xi + u_\eta$

$u_y = u_\xi + u_\eta$

$u_{xx} = 2\,(2u_{\xi\xi} + u_{\xi\eta}) + 2u_{\xi\eta} + u_{\eta\eta}$

$\Rightarrow \quad u_{xx} = 4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta}$

$u_{xy} = 2\,(u_{\xi\xi} + u_{\xi\eta}) + u_{\xi\eta} + u_{\eta\eta} = 2u_{\xi\xi} + 3u_{\xi\eta} + u_{\eta\eta}$

$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$$\begin{aligned} u_{xx} - 3u_{xy} + 2u_{yy} &= 4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta} - 3\,(2u_{\xi\xi} + 3u_{\xi\eta} + u_{\eta\eta}) + 2\,(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \\ &= -u_{\xi\eta} \end{aligned}$$

$\Rightarrow u_{\xi\eta} = 0$

The solution in the original domain is then:

$u(x, y) = f\,(y + 2x) + g\,(y + x)$

1c. $u_{xx} + u_{xy} = 0$

$A = 1 \qquad B = 1 \qquad C = 0 \qquad \triangle = 1 \qquad \underline{\text{hyperbolic}}$

$$\frac{dy}{dx} = \frac{+1 \pm 1}{2} \begin{matrix} \nearrow +1 \\ \searrow 0 \end{matrix}$$

$y = +x + K_1$

$y = K_2$

$$\begin{cases} \xi = y - x & \xi_x = -1 & \xi_y = 1 \\ \eta = y & \eta_x = 0 & \eta_y = 1 \end{cases}$$

$u_x = -u_\xi + u_\eta \underbrace{\eta_x}_{=0} = -u_\xi$

$u_y = u_\xi + u_\eta$

$$u_{xx} = u_{\xi\xi}$$

$$u_{xy} = -u_{\xi\xi} - u_{\xi\eta}$$

$u_{xx} + u_{xy} = -u_{\xi\eta} = 0$

The solution in the original domain is then:
$u = f(y - x) + g(y)$

1d. $u_{xx} + 10u_{xy} + 9u_{yy} = y$

$A = 1 \qquad B = 10 \qquad C = 9 \qquad \triangle = 100 - 36 = 64 \qquad$ <u>hyperbolic</u>

$$\frac{dy}{dx} = \frac{10 \pm 8}{2} \begin{matrix} \nearrow 9 \\ \searrow 1 \end{matrix}$$

$\xi = y - 9x \quad \xi_x = -9 \quad \xi_y = 1$

$\eta = y - x \quad \eta_x = -1 \quad \eta_y = 1$

$u_x = -9u_\xi - u_\eta$

$u_y = u_\xi + u_\eta$

$$\begin{aligned} u_{xx} &= -9\left(-9u_{\xi\xi} - u_{\xi\eta}\right) - \left(-9u_{\xi\eta} - u_{\eta\eta}\right) \\ &= 81u_{\xi\xi} + 18u_{\xi\eta} + u_{\eta\eta} \end{aligned}$$

$$\begin{aligned} u_{xy} &= -9\left(u_{\xi\xi} + u_{\xi\eta}\right) - \left(u_{\xi\eta} + u_{yy}\right) \\ &= -9u_{\xi\xi} - 10u_{\xi\eta} - u_{\eta\eta} \end{aligned}$$

$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$$u_{xx} + 10u_{xy} + 9u_{yy} = \underbrace{(81 - 90 + 9)}_{=0} u_{\xi\xi} + (18 - 100 + 18)u_{\xi\eta} + \underbrace{(1 - 10 + 9)}_{=0} u_{\eta\eta} = y$$

$$-64u_{\xi\eta} = y$$

Substitute for y by using the transformation

$$\left.\begin{aligned} \xi &= y - 9x \\ 9\eta &= 9y - 9x \end{aligned}\right\} -$$

$$\overline{\xi - 9\eta = -8y}$$

$$y = \frac{9\eta - \xi}{8}$$

$$u_{\xi\eta} = \frac{\frac{9\eta - \xi}{8}}{-64} = \frac{\xi}{512} - \frac{9\eta}{512}$$

$$u_{\xi\eta} = \frac{\xi}{512} - \frac{9\eta}{512}$$

To solve this PDE let ξ be fixed and integrate with respect to η

$$\Rightarrow\ u_\xi = \frac{\xi}{512}\,\eta - \frac{9}{512}\,\frac{1}{2}\eta^2 + f\,(\xi)$$

$$u = \frac{1}{2}\,\frac{\xi^2\eta}{512} - \frac{9}{2}\,\frac{1}{512}\,\xi\,\eta^2 + F(\xi) + g\,(\eta)$$

The solution in xy domain is:

$$u(x,\,y) = \frac{(y\,-\,9x)^2(y\,-x)}{1024} - \frac{9}{1024}\,(y\,-\,9x)(y\,-\,x)^2 + F(y\,-\,9x) + g(y\,-\,x)$$

2a. $u_{xx} - u_{yy} + 3u_x - 2u_y + u = 0$

$U = u\,e^{-(\alpha\,\xi + \beta\,\eta)}$

$A = 1 \qquad B = 0 \qquad C = -1 \qquad \triangle = 4 \qquad$ hyperbolic

$$\frac{dy}{dx} = \frac{\pm 2}{2} = \pm 1$$

$\xi = y - x$

$\eta = y + x$

$u_x = -u_\xi + u_\eta$

$u_y = u_\xi + u_\eta$

$u_{xx} = -(-u_{\xi\,\xi} + u_{\xi\,\eta}) + (-u_{\xi\,\eta} + u_{\eta\,\eta}) = u_{\xi\,\xi} - 2u_{\xi\,\eta} + u_{\eta\,\eta}$

$u_{yy} = u_{\xi\,\xi} + 2u_{\xi\,\eta} + u_{\eta\,\eta}$

$-4u_{\xi\,\eta} - 3u_\xi + 3u_\eta - 2u_\xi - 2u_\eta + u = 0$

$-4u_{\xi\,\eta} - 5u_\xi + u_\eta + u = 0$

$U = u\,e^{-(\alpha\,\xi + \beta\,\eta)} \Rightarrow u = U\,e^{(\alpha\,\xi + \beta\,\eta)}$

$u_\xi = U_\xi\,e^{(\alpha\,\xi + \beta\,\eta)} + \alpha\,U\,e^{(\alpha\,\xi + \beta\,\eta)}$

$u_\eta = U_\eta\,e^{(\alpha\,\xi + \beta\,\eta)} + \beta\,U\,e^{(\alpha\,\xi + \beta\,\eta)}$

$u_{\xi\,\eta} = U_{\xi\,\eta}\,e^{(\alpha\,\xi + \beta\,\eta)} + \beta\,U_\xi\,e^{(\alpha\,\xi + \beta\,\eta)} + \alpha\,U_\eta\,e^{(\alpha\,\xi + \beta\,\eta)} + \alpha\,\beta\,U\,e^{(\alpha\,\xi + \beta\,\eta)}$

$-4U_{\xi\,\eta} - 4\beta\,U_\xi - 4\alpha\,U_\eta - 4\alpha\,\beta\,U - 5U_\xi - 5\alpha\,U + U_\eta + \beta\,U + U = 0$

$-4U_{\xi\,\eta} + (-4\beta - 5)U_\xi + (-4\alpha + 1)U_\eta + (-4\alpha\,\beta - 5\alpha + \beta + 1)\,U = 0$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\beta = -5/4 \qquad \alpha = 1/4 \qquad -4(1/4)(-5/4) - 5(1/4) + (-5/4) + 1 = -1/4$$

$$-4U_{\xi\,\eta} - \frac{1}{4}U = 0$$

$\boxed{U_{\xi\,\eta} = -\frac{1}{16}\,U}$ required form

2b. $3u_{xx} + 7u_{xy} + 2u_{yy} + u_y + u = 0$

$A = 3 \qquad B = 7 \qquad C = 2 \qquad \triangle = 49 - 24 = 25 \qquad$ <u>hyperbolic</u>

$$\frac{dy}{dx} = \frac{7 \pm 5}{6} \begin{array}{l} \nearrow 2 \\ \searrow \frac{1}{3} \end{array}$$

$\xi = y - 2x \qquad \xi_x = -2 \qquad \xi_y = 1$

$\eta = y - \dfrac{1}{3}x \qquad \eta_x = -\frac{1}{3} \qquad \eta_y = 1$

$$u_x = -2u_\xi - \frac{1}{3}u_\eta$$

$$u_y = u_\xi + u_\eta$$

$$u_{xx} = -2\left(-2u_{\xi\xi} - \frac{1}{3}u_{\xi\eta}\right) - \frac{1}{3}\left(-2u_{\xi\eta} - \frac{1}{3}u_{\eta\eta}\right)$$

$$u_{xx} = 4u_{\xi\xi} + \frac{4}{3}u_{\xi\eta} + \frac{1}{9}u_{\eta\eta}$$

$$u_{xy} = -2\left(u_{\xi\xi} + u_{\xi\eta}\right) - \frac{1}{3}\left(u_{\xi\eta} + u_{\eta\eta}\right)$$

$$u_{xy} = -2u_{\xi\xi} - \frac{7}{3}u_{\xi\eta} - \frac{1}{3}u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$4u_{\xi\eta} - \frac{49}{3}u_{\xi\eta} + 4u_{\xi\eta} + u_\xi + u_\eta + u = 0$$

$$\boxed{-\tfrac{25}{3}u_{\xi\eta} + u_\xi + u_\eta + u = 0}$$

Use last page:

$$\frac{-25}{3}\left(U_{\xi\eta} + \beta U_\xi + \alpha U_\eta + \alpha\beta U\right) + U_\xi + \alpha U + U_\eta + \beta U + U = 0$$

$$\frac{-25}{3}U_{\xi\eta} + \underbrace{\left(\frac{-25}{3}\beta + 1\right)}_{\Downarrow}U_\xi + \underbrace{\left(\frac{-25}{3}\alpha + 1\right)}_{\Downarrow}U_\eta + \underbrace{\left(\frac{-25}{3}\alpha\beta + \alpha + \beta + 1\right)}_{\Downarrow}U = 0$$

$$\beta = 3/25 \qquad\qquad \alpha = 3/25 \qquad\qquad -\frac{3}{25} + \frac{3}{25} + \frac{3}{25} + 1 = \frac{28}{25}$$

$$\frac{-25}{3}u_{\xi\eta} + \frac{28}{25}U = 0 \qquad \Rightarrow \qquad \boxed{U_{\xi\eta} = \tfrac{3}{25}\,\tfrac{28}{25}U}$$

3.

$$u_{xx} = au_t + bu_x - \frac{b^2}{4}u + d$$

$A = 1 \qquad B = C = 0 \qquad \Rightarrow \qquad \triangle = 0$ <u>parabolic</u>

$\frac{dx}{dt} = 0$ already in canonical form since u_{xx} is the only 2^{nd} order term

$$u = \upsilon e^{\frac{b}{2}x}$$

$$u_x = \upsilon_x e^{\frac{b}{2}x} + \frac{b}{2}\upsilon e^{\frac{b}{2}x}$$

$$u_{xx} = \upsilon_{xx}e^{\frac{b}{2}x} + b\,\upsilon_x e^{\frac{b}{2}x} + \frac{b^2}{4}\upsilon e^{\frac{b}{2}x}$$

$$u_t = \upsilon_t e^{\frac{b}{2}x}$$

$$\Rightarrow \quad \upsilon_{xx} + b\upsilon_x + \frac{b^2}{4}\upsilon = a\upsilon_t + b\left(\upsilon_x + \frac{b}{2}\upsilon\right) - \frac{b^2}{4}\upsilon + de^{-\frac{b}{2}x}$$

Since υ_x and υ terms cancel out we have:

$$\upsilon_{xx} = a\upsilon_t + de^{-\frac{b}{2}x}$$

CHAPTER 3

3 Method of Characteristics

3.1 Advection Equation (first order wave equation)

Problems

1. Solve
$$\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = 0$$
subject to
$$w(x,0) = \sin x$$

2. Solve using the method of characteristics

   a. $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = e^{2x}$ subject to $u(x,0) = f(x)$

   b. $\dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = 1$ subject to $u(x,0) = f(x)$

   c. $\dfrac{\partial u}{\partial t} + 3t\dfrac{\partial u}{\partial x} = u$ subject to $u(x,0) = f(x)$

   d. $\dfrac{\partial u}{\partial t} - 2\dfrac{\partial u}{\partial x} = e^{2x}$ subject to $u(x,0) = \cos x$

   e. $\dfrac{\partial u}{\partial t} - t^2\dfrac{\partial u}{\partial x} = -u$ subject to $u(x,0) = 3e^{x}$

3. Show that the characteristics of
$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$
$$u(x,0) = f(x)$$
are straight lines.

4. Consider the problem
$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$
$$u(x,0) = f(x) = \begin{cases} 1 & x < 0 \\ 1 + \frac{x}{L} & 0 < x < L \\ 2 & L < x \end{cases}$$

   a. Determine equations for the characteristics

   b. Determine the solution $u(x,t)$

   c. Sketch the characteristic curves.

   d. Sketch the solution $u(x,t)$ for fixed t.

1. The PDE can be rewrriten as a system of two ODEs

$$\frac{dx}{dt} = -3$$

$$\frac{dw}{dt} = 0$$

The solution of the first gives the characteristic curve

$$x + 3t = x_0$$

and the second gives

$$w(x(t), t) = w(x(0), 0) = \sin x_0 = \sin(x + 3t)$$

$\boxed{w(x, t) = \sin(x + 3t)}$

2. a. The ODEs in this case are

$$\frac{dx}{dt} = c$$

$$\frac{du}{dt} = e^{2x}$$

Solve the characteristic equation

$$x = ct + x_0$$

Now solve the second ODE. To do that we have to plug in for x

$$\frac{du}{dt} = e^{2(x_0 + ct)} = e^{2x_0}\, e^{2ct}$$

$$u(x, t) = e^{2x_0}\, \frac{1}{2c}\, e^{2ct} + K$$

The constant of integration can be found from the initial condition

$$f(x_0) = u(x_0, 0) = \frac{1}{2c}\, e^{2x_0} + K$$

Therefore

$$K = f(x_0) - \frac{1}{2c}\, e^{2x_0}$$

Plug this K in the solution

$$u(x, t) = \frac{1}{2c}\, e^{2x_0 + 2ct} + f(x_0) - \frac{1}{2c}\, e^{2x_0}$$

Now substitute for x_0 from the characteristic curve $\boxed{u(x, t) = \frac{1}{2c}\, e^{2x} + f(x - ct) - \frac{1}{2c}\, e^{2(x - ct)}}$

2. b. The ODEs in this case are

$$\frac{dx}{dt} = x$$
$$\frac{du}{dt} = 1$$

Solve the characteristic equation

$$\ln x = t + \ln x_0 \qquad \text{or} \qquad x = x_0 \, e^t$$

The solution of the second ODE is

$$u = t + K \qquad \text{and the constant is} \qquad f(x_0)$$

$$u(x,t) = t + f(x_0)$$

Substitute x_0 from the characteristic curve $\boxed{u(x,t) = t + f\left(x \, e^{-t}\right)}$

2. c. The ODEs in this case are

$$\frac{dx}{dt} = 3t$$
$$\frac{du}{dt} = u$$

Solve the characteristic equation

$$x = \frac{3}{2} t^2 + x_0$$

The second ODE can be written as

$$\frac{du}{u} = dt$$

Thus the solution of the second ODE is

$$\ln u = t + \ln K \qquad \text{and the constant is} \qquad f(x_0)$$

$$u(x,t) = f(x_0) \, e^t$$

Substitute x_0 from the characteristic curve $\boxed{u(x,t) = f\left(x - \frac{3}{2} t^2\right) \, e^t}$

2.d. The ODEs in this case are

$$\frac{dx}{dt} = -2$$
$$\frac{du}{dt} = e^{2x}$$

Solve the characteristic equation

$$x = -2t + x_0$$

Now solve the second ODE. To do that we have to plug in for x

$$\frac{du}{dt} = e^{2(x_0 - 2t)} = e^{2x_0}\, e^{-4t}$$

$$u(x,t) = e^{2x_0}\left(-\frac{1}{4}\, e^{-4t}\right) + K$$

The constant of integration can be found from the initial condition

$$\cos(x_0) = u(x_0, 0) = -\frac{1}{4}\, e^{2x_0} + K$$

Therefore

$$K = \cos(x_0) + \frac{1}{4}\, e^{2x_0}$$

Plug this K in the solution and substitute for x_0 from the characteristic curve

$$u(x,t) = -\frac{1}{4}\, e^{2(x+2t)}\, e^{-4t} + \cos(x + 2t) + \frac{1}{4}\, e^{2(x+2t)}$$

$$\boxed{u(x,t) = \frac{1}{4}\, e^{2x}\left(e^{4t} - 1\right) + \cos(x + 2t)}$$

To check the answer, we differentiate

$$u_x = \frac{1}{2}\, e^{2x}\left(e^{4t} - 1\right) - \sin(x + 2t)$$

$$u_t = \frac{1}{4}\, e^{2x}\left(4\, e^{4t}\right) - 2\sin(x + 2t)$$

Substitute in the PDE

$$u_t - 2u_x = e^{2x}\, e^{4t} - 2\sin(x + 2t) - 2\left\{\frac{1}{2}\, e^{2x}\left(e^{4t} - 1\right) - \sin(x + 2t)\right\}$$

$$= e^{2x}\, e^{4t} - 2\sin(x + 2t) - e^{2x}\, e^{4t} + e^{2x} + 2\sin(x + 2t)$$

$$= e^{2x} \qquad \text{which is the right hand side of the PDE}$$

2. e. The ODEs in this case are

$$\frac{dx}{dt} = -t^2$$

$$\frac{du}{dt} = -u$$

Solve the characteristic equation

$$x = -\frac{t^3}{3} + x_0$$

Now solve the second ODE. To do that we rewrite it as

$$\frac{du}{u} = -dt$$

Therefore the solution as in 2c

$$\ln u = -t + \ln K \qquad \text{and the constant is} \qquad 3\,e^{x_0}$$

Plug this K in the solution and substitute for x_0 from the characteristic curve

$$\ln u(x,t) = \ln\left[3\,e^{x+\frac{1}{3}t^3}\right] - t$$

$$\boxed{u(x,t) = 3\,e^{x+\frac{1}{3}t^3}\,e^{-t}}$$

To check the answer, we differentiate

$$u_t = 3\,e^x\left(t^2 - 1\right)e^{\frac{1}{3}t^3 - t}$$

$$u_x = 3\,e^x\,e^{\frac{1}{3}t^3 - t}$$

Substitute in the PDE

$$u_t - t^2 u_x = 3\,e^x\,e^{\frac{1}{3}t^3 - t} - t^2\left\{3\,e^x\left(t^2 - 1\right)e^{\frac{1}{3}t^3 - t}\right\}$$

$$= 3\,e^x\,e^{\frac{1}{3}t^3 - t}\left[\left(t^2 - 1\right) - t^2\right] = -3\,e^{x+\frac{1}{3}t^3 - t} = -u$$

3. The ODEs in this case are

$$\frac{dx}{dt} = 2u$$

$$\frac{du}{dt} = 0$$

Since the first ODE contains x, t and u, we solve the second ODE first

$$u(x,t) = u(x(0),0) = f(x(0))$$

Plug this u in the first ODE, we get

$$\frac{dx}{dt} = 2f(x(0))$$

The solution is

$$x = x_0 + 2tf(x_0)$$

These are characteristics lines all with slope

$$\frac{1}{2f(x_0)}$$

Note that the characteristic through $x_1(0)$ will have a different slope than the one through $x_2(0)$. That is the straight line are NOT parallel.

4. The ODEs in this case are

$$\frac{dx}{dt} = 2u$$
$$\frac{du}{dt} = 0$$

with

$$u(x,0) = f(x) = \begin{cases} 1 & x < 0 \\ 1 + \frac{x}{L} & 0 < x < L \\ 2 & L < x \end{cases}$$

a. Since the first ODE contains x, t and u, we solve the second ODE first

$$u(x,t) = u(x(0),0) = f(x(0))$$

Plug this u in the first ODE, we get

$$\frac{dx}{dt} = 2f(x(0))$$

The solution is

$$x = x_0 + 2tf(x_0)$$

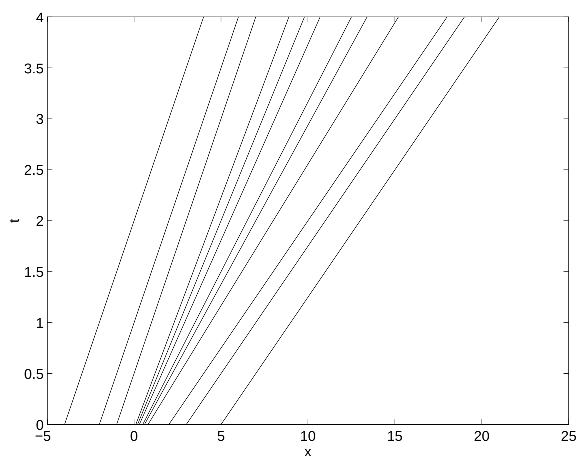

Figure 12: Characteristics for problem 4

b. For $x_0 < 0$ then $f(x_0) = 1$ and the solution is

$$u(x,t) = 1 \qquad \text{on } x = x_0 + 2t$$

or

$$u(x,t) = 1 \qquad \text{on } x < 2t$$

For $x_0 > L$ then $f(x_0) = 2$ and the solution is

$$u(x,t) = 2 \qquad \text{on } x > 4t + L$$

For $0 < x_0 < L$ then $f(x_0) = 1 + x_0/L$ and the solution is

$$u(x,t) = 1 + \frac{x_0}{L} \qquad \text{on } x = 2t\left(1 + \frac{x_0}{L}\right) + x_0$$

That is

$$x_0 = \frac{x - 2t}{2t + L} L$$

Thus the solution on this interval is

$$u(x,t) = 1 + \frac{x - 2t}{2t + L} = \frac{2t + L + x - 2t}{2t + L} = \frac{x + L}{2t + L}$$

Notice that u is continuous.

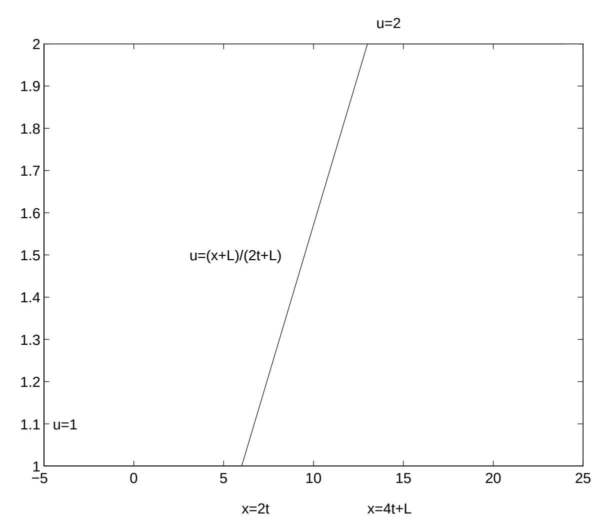

Figure 13: Solution for problem 4

3.2 Quasilinear Equations

3.2.1 The Case $S = 0,\ c = c(u)$

Problems

1. Solve the following

   a. $\dfrac{\partial u}{\partial t} = 0 \qquad$ subject to $u(x,0) = g(x)$

   b. $\dfrac{\partial u}{\partial t} = -3xu \qquad$ subject to $u(x,0) = g(x)$

2. Solve

$$\frac{\partial u}{\partial t} = u$$

subject to

$$u(x,t) = 1 + \cos x \qquad \text{along} \qquad x + 2t = 0$$

3. Let

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \qquad c = \text{constant}$$

   a. Solve the equation subject to $u(x,0) = \sin x$

   b. If $c > 0$, determine $u(x,t)$ for $x > 0$ and $t > 0$ where

$$\begin{aligned} u(x,0) &= f(x) \qquad \text{for } x > 0 \\ u(0,t) &= g(t) \qquad \text{for } t > 0 \end{aligned}$$

4. Solve the following linear equations subject to $u(x,0) = f(x)$

   a. $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = e^{-3x}$ b. $\dfrac{\partial u}{\partial t} + t\dfrac{\partial u}{\partial x} = 5$

   c. $\dfrac{\partial u}{\partial t} - t^2\dfrac{\partial u}{\partial x} = -u$

   d. $\dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = t$

   e. $\dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = x$

5. Determine the parametric representation of the solution satisfying $u(x,0) = f(x)$,

   a. $\dfrac{\partial u}{\partial t} - u^2\dfrac{\partial u}{\partial x} = 3u$

b. $\dfrac{\partial u}{\partial t} + t^2 u \dfrac{\partial u}{\partial x} = -u$

6. Solve

$$\frac{\partial u}{\partial t} + t^2 u \frac{\partial u}{\partial x} = 5$$

subject to

$$u(x, 0) = x.$$

1.

a. Integrate the PDE assuming x fixed, we get

$$u(x,t) = K(x)$$

Since $dx/dt = 0$ we have $x = x_0$ and thus

$$u(x,t) = u(x_0, 0) = K(x_0) = g(x_0) = g(x)$$
$$u(x,t) = g(x)$$

b. For a fixed x, we can integrate the PDE with respect to t

$$\int \frac{du}{u} = -3xt + K(x)$$

$$\ln u - \ln c(x) = -3xt$$
$$u(x,t) = ce^{-3xt}$$

Using the initial condition

$$u(x,t) = f(x)\, e^{-3xt}$$

2. The set of ODEs are

$$\frac{dx}{dt} = 0 \qquad \text{and } \frac{du}{dt} = u$$

The characteristics are $x = $ constant and the ODE for u can be written

$$\frac{du}{u} = dt$$

Thus

$$u(x,t) = k(x)\, e^t$$

On $x = -2t$ or $x + 2t = 0$ we have

$$1 + \cos x = k(x)\, e^t|_{x=-2t} = k(x)\, e^{-\frac{x}{2}}$$

Thus the constant of integration is

$$k(x) = e^{\frac{x}{2}}\ (1 + \cos x)$$

Plug this in the solution u we get

$$u(x,t) = (1 + \cos x)\ e^{\frac{x}{2}+t}$$

Another way of getting the solution is by a rotation so that the line $x + 2t = 0$ becomes horizontal. Call that axis ξ, the line perpendicular to it is given by $t - 2x = 0$, which we call η.

So here is the transformation

$$\xi = x + 2t$$

$$\eta = t - 2x.$$

The PDE becomes:

$$u_\xi + \frac{1}{2} u_\eta = \frac{1}{2} u$$

and the intial condition is:

$$u(\eta, \xi = 0) = 1 + \cos \frac{\xi - 2\eta}{5}|_{\xi=0} = 1 + \cos \frac{2}{5}\eta$$

Rewrite this as a system of two first order ODEs,

$$\frac{d\eta}{d\xi} = \frac{1}{2}$$

$$\eta(0) = \alpha$$

$$\frac{du(\eta(\xi), \xi)}{d\xi} = \frac{1}{2} u$$

$$u(\eta(0),0) \;=\; 1+\cos\frac{2}{5}\alpha.$$

The solution of the first ODE, gives the characteristics in the transformed domain:

$$\eta \;=\; \frac{1}{2}\xi+\alpha$$

The solution of the second ODE:

$$u(\eta(\xi),\xi) \;=\; Ke^{\frac{1}{2}\xi}$$

Using the initial condition

$$1+\cos\frac{2}{5}\alpha \;=\; K$$

Thus

$$u(\eta(\xi),\xi) \;=\; (1+\cos\frac{2}{5}\alpha)e^{\frac{1}{2}\xi}$$

But $\alpha = \eta - \frac{1}{2}\xi$ thus

$$u(\eta(\xi),\xi) \;=\; (1+\cos\frac{2}{5}(\eta-\frac{1}{2}\xi))e^{\frac{1}{2}\xi}$$

Now substitute back:

$$\frac{1}{2}\xi \;=\; \frac{1}{2}x+t$$

$$\eta-\frac{1}{2}\xi \;=\; (t-2x)-(\frac{1}{2}x+t) \;=\; \frac{5}{2}x$$

Thus

$$u(x,t) \;=\; (1+\cos x)e^{\frac{1}{2}x+t}.$$

3. a. The set of ODEs to solve is

$$\frac{dx}{dt} = c \qquad\qquad \frac{du}{dt} = 0$$

The characteristics are:

$$x = x_0 + ct$$

The solution of the second ODE is

$$u(x,t) = \text{constant} = u(x_0, 0) = \sin x_0$$

Substitute for x_0, we get

$$u(x,t) = \sin(x - ct)$$

b. For $x > ct$ the solution is $u(x,t) = f(x - ct)$

But $f(x)$ is defined only for positive values of the independent variable x, therefore $f(x - ct)$ cannot be used for $x < ct$.

In this case $(x < ct)$ we must use the condition

$$u(0,t) = g(t)$$

The characteristics for which $x_0 < 0$ is given by $x = x_0 + ct$ and it passes through the point $(0, t_0)$ (see figure). Thus $x = c(t - t_0)$ and $u(0, t_0) = g(t_0) = g\left(t - \frac{x}{c}\right)$

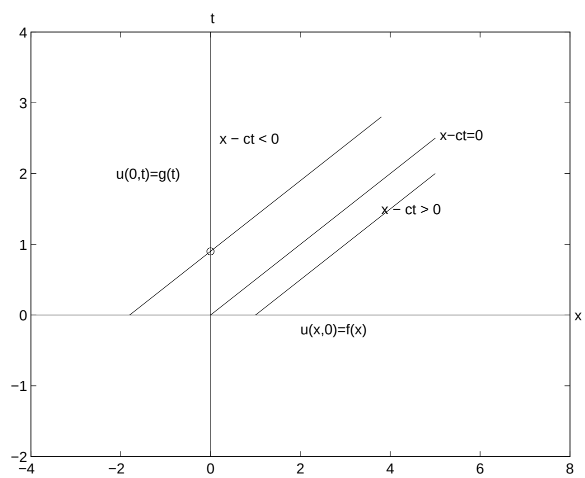

Figure 14: Domain and characteristics for problem 3b

The solution is therefore given by

$$u(x,t) = \begin{cases} f(x - ct) & \text{for } x - ct > 0 \\ g\left(t - \frac{x}{c}\right) & \text{for } x - ct < 0 \end{cases}$$

4. a. The set of ODEs is

$$\frac{dx}{dt} = c \qquad\qquad \frac{du}{dt} = e^{-3x}$$

The solution of the first is

$$x = x_0 + ct$$

Substituting x in the second ODE

$$\frac{du}{dt} = e^{-3(x_0+ct)}$$

Now integrate

$$u(x,t) = K + e^{-3x_0} \frac{1}{-3c} e^{-3ct}$$

At $t = 0$ we get

$$f(x_0) = u(x_0, 0) = K + e^{-3x_0} \frac{1}{-3c}$$

Therefore the constant of integration K is

$$K = f(x_0) + e^{-3x_0} \frac{1}{3c}$$

Substitute this K in the solution

$$u(x,t) = f(x_0) + e^{-3x_0} \frac{1}{3c} - e^{-3x_0} \frac{1}{3c} e^{-3ct}$$

Recall that $x_0 = x - ct$ thus

$$u(x,t) = f(x - ct) + \frac{1}{3c} e^{-3(x-ct)} - \frac{1}{3c} e^{-3x}$$

b. The set of ODEs is

$$\frac{dx}{dt} = t \qquad\qquad \frac{du}{dt} = 5$$

The solution of the first is

$$x = x_0 + \frac{1}{2}t^2$$

Now integrate the second ODE

$$u(x,t) = 5t + K$$

At $t = 0$ the solution is

$$u(x_0, 0) = f(x_0) = K \qquad \text{plug } t = 0 \text{ in the solution } u$$

Thus when substituting for x_0 in the solution

$$u(x,t) = 5t + f\left(x - \frac{1}{2}t^2\right)$$

c. The set of ODEs is

$$\frac{dx}{dt} = -t^2 \qquad\qquad \frac{du}{dt} = u$$

The solution of the first is

$$x = x_0 - \frac{1}{3}t^3$$

Now integrate the second ODE

$$\ln u(x,t) = -t + \ln K$$

or

$$u(x,t) = K\, e^{-t}$$

At $t = 0$ the solution is

$$u(x_0, 0) = f(x_0) = K \qquad \text{plug } t = 0 \text{ in the solution } u$$

Thus when substituting for x_0 in the solution

$$u(x,t) = e^{-t} f\left(x + \frac{1}{3}t^3\right)$$

d. The set of ODEs is

$$\frac{dx}{dt} = x \qquad\qquad \frac{du}{dt} = t$$

The solution of the first is

$$\ln x = \ln x_0 + t$$

or

$$x = x_0\, e^t$$

Now integrate the second ODE

$$u(x,t) = \frac{1}{2}t^2 + K$$

At $t = 0$ the solution is

$$u(x_0, 0) = f(x_0) = K \qquad \text{plug } t = 0 \text{ in the solution } u$$

Thus when substituting for x_0 in the solution

$$u(x,t) = \frac{1}{2}t^2 + f\left(x\, e^{-t}\right)$$

e. The set of ODEs is

$$\frac{dx}{dt} = x \qquad\qquad \frac{du}{dt} = x$$

The solution of the first is

$$\ln x = \ln x_0 + t$$

or

$$x = x_0 \, e^t$$

Now substitute x in the second ODE

$$\frac{du}{dt} = x_0 \, e^t$$

and integrate it

$$u(x,t) = e^t \, x_0 + K$$

At $t = 0$ the solution is

$$u(x_0, 0) = f(x_0) = K + x_0 \qquad \text{plug } t = 0 \text{ in the solution } u$$

Thus when substituting K in u

$$u(x,t) = x_0 \, e^t + f(x_0) - x_0$$

Now substitute for x_0 in the solution

$$u(x,t) = x + f\left(x \, e^{-t}\right) - x \, e^{-t}$$

5. a. The set of ODEs is

$$\frac{dx}{dt} = -u^2 \qquad\qquad \frac{du}{dt} = 3u$$

The solution of the first ODE requires the yet unknown u thus we tackle the second ODE

$$\frac{du}{u} = 3\,dt$$

Now integrate this

$$\ln u(x,t) = 3t + K \qquad\qquad \text{or } u(x,t) = C\,e^{3t}$$

At $t = 0$ the solution is

$$u(x_0, 0) = f(x_0) = C$$

Thus

$$u(x,t) = f(x_0)\,e^{3t}$$

Now substitute this solution in the characteristic equation (first ODE)

$$\frac{dx}{dt} = -\left(f(x_0)\,e^{3t}\right)^2 = -\left(f(x_0)\right)^2\,e^{6t}$$

Integrating

$$x = -\left(f(x_0)\right)^2 \int e^{6t} dt = -\frac{1}{6}\left(f(x_0)\right)^2 e^{6t} + K$$

For $t = 0$ we get

$$x_0 = -\frac{1}{6}\left(f(x_0)\right)^2 + K$$

Thus

$$K = x_0 + \frac{1}{6}\left(f(x_0)\right)^2$$

and the characteristics are

$$x = -\frac{1}{6}\left(f(x_0)\right)^2\,e^{6t} + x_0 + \frac{1}{6}\left(f(x_0)\right)^2$$

"Solve" this for x_0 and subsitute for u. The quote is because one can only solve this for special cases of the function $f(x_0)$.

The implicit solution is given by $$\boxed{\begin{aligned} u(x,t) &= f(x_0)\,e^{3t} \\ x &= -\tfrac{1}{6}\left(f(x_0)\right)^2\,e^{6t} + x_0 + \tfrac{1}{6}\left(f(x_0)\right)^2 \end{aligned}}$$

b. The set of ODEs is

$$\frac{dx}{dt} = t^2\, u \qquad\qquad \frac{du}{dt} = -u$$

The solution of the first ODE requires the yet unknown u thus we tackle the second ODE

$$\frac{du}{u} = -\,dt$$

Now integrate this

$$\ln u(x,t) = -t + K \qquad\qquad \text{or } u(x,t) = C\, e^{-t}$$

At $t = 0$ the solution is

$$u(x_0, 0) = f(x_0) = C$$

Thus

$$u(x,t) = f(x_0)\, e^{-t}$$

Now substitute this solution in the characteristic equation (first ODE)

$$\frac{dx}{dt} = t^2\, f(x_0)\, e^{-t}$$

or

$$\int dx = f(x_0) \int t^2\, e^{-t}\, dt$$

Integrate and continue as in part a of this problem

$$x = f(x_0)\left[-t^2\, e^{-t} - 2t\, e^{-t} - 2\, e^{-t} + C\right]$$

For $t = 0$ we get

$$x_0 = f(x_0)\,[-2 + C]$$

Thus

$$C\, f(x_0) = x_0 + 2\, f(x_0)$$

and the characteristics are

$$x = f(x_0)\left[-t^2 - 2t - 2\right]\, e^{-t} + x_0 + 2\, f(x_0)$$

"Solve" this for x_0 and subsitute for u. The quote is because one can only solve this for special cases of the function $f(x_0)$.

The implicit solution is given by
$$\boxed{\begin{aligned} u(x,t) &= f(x_0)\, e^{-t} \\ x &= -\, f(x_0)\,[\,t^2 + 2t + 2\,]\; e^{-t} + x_0 + 2\, f(x_0) \end{aligned}}$$

6. The set of ODEs is
$$\frac{dx}{dt} = t^2\, u \qquad\qquad \frac{du}{dt} = 5$$
The solution of the first ODE requires the yet unknown u thus we tackle the second ODE
$$du = 5\, dt$$
Now integrate this
$$u(x,t) = 5t + K$$
At $t = 0$ the solution is
$$u(x_0, 0) = f(x_0) = x_0 = K$$
Thus
$$u(x,t) = x_0 + 5t$$
Now substitute this solution in the characteristic equation (first ODE)
$$\frac{dx}{dt} = 5t^3 + x_0\, t^2$$
Integrate
$$x = \frac{5}{4}\, t^4 + \frac{1}{3}\, t^3\, x_0 + C$$
For $t = 0$ we get
$$x_0 = 0 + 0 + C$$
Thus
$$C = x_0$$
and the characteristics are
$$x = \frac{5}{4}\, t^4 + \left(\frac{1}{3}\, t^3 + 1\right) x_0$$
Solve this for x_0
$$x_0 = \frac{x - \frac{5}{4}\, t^4}{1 + \frac{1}{3}\, t^3}$$
The solution is then given by $\boxed{u(x,t) = 5t + \frac{x - \frac{5}{4}\, t^4}{1 + \frac{1}{3}\, t^3}}$

3.2.3 Fan-like Characteristics

3.2.4 Shock Waves

Problems

1. Consider Burgers' equation

$$\frac{\partial \rho}{\partial t} + u_{max}\left[1 - \frac{2\rho}{\rho_{max}}\right]\frac{\partial \rho}{\partial x} = \nu \frac{\partial^2 \rho}{\partial x^2}$$

Suppose that a solution exists as a density wave moving without change of shape at a velocity V, $\rho(x,t) = f(x - Vt)$.

a. What ordinary differential equation is satisfied by f

b. Show that the velocity of wave propagation, V, is the same as the shock velocity separating $\rho = \rho_1$ from $\rho = \rho_2$ (occuring if $\nu = 0$).

2. Solve

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial \rho}{\partial x} = 0$$

subject to

$$\rho(x,0) = \begin{cases} 4 & x < 0 \\ 3 & x > 0 \end{cases}$$

3. Solve

$$\frac{\partial u}{\partial t} + 4u\frac{\partial u}{\partial x} = 0$$

subject to

$$u(x,0) = \begin{cases} 3 & x < 1 \\ 2 & x > 1 \end{cases}$$

4. Solve the above equation subject to

$$u(x,0) = \begin{cases} 2 & x < -1 \\ 3 & x > -1 \end{cases}$$

5. Solve the quasilinear equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

subject to

$$u(x,0) = \begin{cases} 2 & x < 2 \\ 3 & x > 2 \end{cases}$$

6. Solve the quasilinear equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

subject to

$$u(x,0) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$

7. Solve the inviscid Burgers' equation

$$u_t + uu_x = 0$$

$$u(x,0) = \begin{cases} 2 & \text{for} \quad x < 0 \\ \\ 1 & \text{for} \quad 0 < x < 1 \\ \\ 0 & \text{for} \quad x > 1 \end{cases}$$

Note that two shocks start at $t = 0$, and eventually intersect to create a third shock. Find the solution for all time (analytically), and graphically display your solution, labeling all appropriate bounding curves.

1. a. Since

$$\rho(x,t) = f(x - Vt)$$

we have (using the chain rule)

$$\rho_t = f'(x - Vt) \cdot (-V)$$
$$\rho_x = f'(x - Vt) \cdot 1$$
$$\rho_{xx} = f''(x - Vt)$$

Substituting these derivatives in the PDE we have

$$-V\,f'(x - Vt) + u_{max}\left(1 - \frac{2f(x - Vt)}{\rho_{max}}\right) f'(x - Vt) = \nu\,f''(x - Vt)$$

This is a second order ODE for f.

b. For the case $\nu = 0$ the ODE becomes

$$-V\,f'(x - Vt) + u_{max}\left(1 - \frac{2f(x - Vt)}{\rho_{max}}\right) f'(x - Vt) = 0$$

Integrate (recall that the integral of $2ff'$ is f^2)

$$-V\,f(x - Vt) + u_{max}\left(f(x - Vt) - \frac{f^2(x - Vt)}{\rho_{max}}\right) = C$$

To find the constant, we use the following
As $x \to \infty$, $\rho \to \rho_2$ and as $x \to -\infty$, $\rho \to \rho_1$, then

$$-V\,\rho_2 + u_{max}\left(\rho_2 - \frac{\rho_2^2}{\rho_{max}}\right) = C$$

$$-V\,\rho_1 + u_{max}\left(\rho_1 - \frac{\rho_1^2}{\rho_{max}}\right) = C$$

Subtract

$$V\,(\rho_1 - \rho_2) + u_{max}\left(\rho_2 - \frac{\rho_2^2}{\rho_{max}}\right) - u_{max}\left(\rho_1 - \frac{\rho_1^2}{\rho_{max}}\right) = 0$$

Solve for V

$$V = \frac{u_{max}\left(\rho_2 - \frac{\rho_2^2}{\rho_{max}}\right) - u_{max}\left(\rho_1 - \frac{\rho_1^2}{\rho_{max}}\right)}{\rho_2 - \rho_1} \tag{1}$$

This can be written as

$$V = u_{max} - \frac{u_{max}}{\rho_{max}}(\rho_1 + \rho_2)$$

Note that (1) is

$$V = \frac{[q]}{[\rho]}$$

since

$$q = u_{max}\left(\rho - \frac{\rho^2}{\rho_{max}}\right)$$

Thus V given in (1) is exactly the shock speed.

2. The set of ODEs is

$$\frac{dx}{dt} = \rho^2 \qquad \frac{d\rho}{dt} = 0$$

The solution of the first ODE requires the yet unknown ρ thus we tackle the second ODE

$$d\rho = 0$$

Now integrate this

$$\rho(x,t) = K$$

At $t = 0$ the solution is

$$\rho(x_0, 0) = K = \begin{cases} 4 & x < 0 \\ 3 & x > 0 \end{cases}$$

Thus

$$\rho(x,t) = \rho(x_0, 0)$$

Now substitute this solution in the characteristic equation (first ODE)

$$\frac{dx}{dt} = \rho^2(x_0, 0)$$

Integrate

$$x = \rho^2(x_0, 0)\, t + C$$

For $t = 0$ we get

$$x_0 = 0 + C$$

Thus

$$C = x_0$$

and the characteristics are

$$x = \rho^2(x_0, 0)\, t + x_0$$

For $x_0 < 0$ then $\rho(x_0, 0) = 4$ and the characteristic is then given by $x = x_0 + 16t$
Therefore for $x_0 = x - 16t < 0$ the solution is $\rho = 4$.

For $x_0 > 0$ then $\rho(x_0, 0) = 3$ and the characteristic is then given by $x = x_0 + 9t$
Therefore for $x_0 = x - 9t < 0$ the solution is $\rho = 3$.

Notice that there is a shock (since the value of ρ is decreasing with increasing x). The shock characteristc is given by

$$\frac{dx_s}{dt} = \frac{\frac{1}{3} \cdot 4^3 - \frac{1}{3} \cdot 3^3}{4 - 3} = \frac{\frac{1}{3}(64 - 27)}{1} = \frac{37}{3}$$

The solution of this ODE is

$$x_s = \frac{37}{3} t + x_s(0)$$

$x_s(0)$ is where the shock starts, i.e. the discontinuity at time $t = 0$.

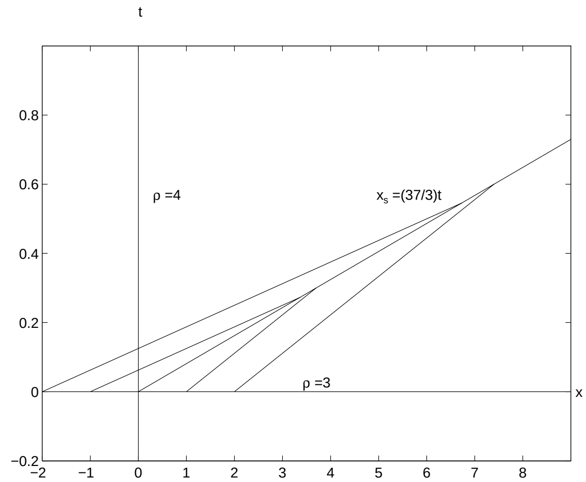

Figure 15: Characteristics for problem 2

Thus $x_s(0) = 0$ and the shock characteristic is

$$x_s = \frac{37}{3}t$$

See figure for the characteristic curves including the shock's. The solution in region I above the shock chracteristic is $\rho = 4$ and below (region II) is $\rho = 3$.

3.

$$u_t + 4uu_x = 0 \qquad\qquad \text{or } u_t + (2u^2)_x = 0$$

$$u(x,0) = \begin{cases} 3 & x < 1 \\ 2 & x > 1 \end{cases}$$

Shock again

The shock characteristic is obtained by solving:

$$\frac{dx_s}{dt} = \frac{2 \cdot 3^2 - 2 \cdot 2^2}{3 - 2} = 10$$

$$x_s = 10t + \underbrace{x_s(0)}_{=1}$$

$$\underline{x_s = 10t + 1}$$

Now we solve the ODE for u:

$$\frac{du}{dt} = 0 \quad \Rightarrow \quad \underline{u(x,t) = u(x_0, 0)} \qquad \text{away from shock}$$

The ODE for x is:

$$\frac{dx}{dt} = 4u = 4u(x_0, 0)$$

$$\underline{x = 4u(x_0, 0))t + x_0}$$

If $\quad x_0 < 1 \qquad x_0 = x - 12t \qquad \Rightarrow \qquad \underline{x < 1 + 12t}$

$\quad x_0 < 1 \qquad x_0 = x - 8t \qquad \Rightarrow \qquad \underline{x > 1 + 8t}$

4. Solve

$$u_t + 4uu_x = 0$$

$$u(x, 0) = \begin{cases} 2 & x < -1 \\ \\ 3 & x > -1 \end{cases}$$

$$\frac{du}{dt} = 0 \qquad u(x, t) = u(x_0, 0)$$

$$\frac{dx}{dt} = 4u = 4u(x_0, 0)$$

$$dx = 4u\,(x_0, 0)\,dt$$

$$\underline{x = 4u\,(x_0, 0)\,t + x_0}$$

For $\quad x_0 < -1 \qquad x = 8t + x_0 \qquad \Rightarrow \qquad x - 8t < -1$

$x_0 > -1 \qquad x = 12t + x_0 \qquad \Rightarrow \qquad x - 12t > -1$

$$u(x, t) = \begin{cases} 2 & x < 8t - 1 \\ \\ ? & 8t - 1 < x < 12t - 1 \\ \\ 3 & x > 12t - 1 \end{cases}$$

$$x = 4u\,t + \underbrace{x_0}_{= -1 \text{ discontinuity}}$$

$$u = \frac{x + 1}{4t}$$

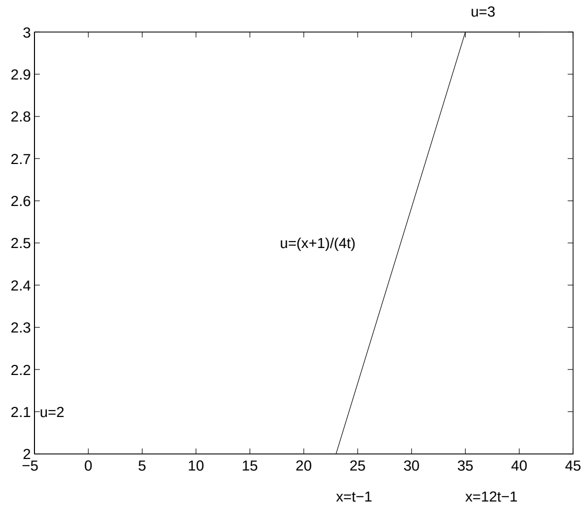

Figure 16: Solution for 4

5.

$$u_t + uu_x = 0$$

$$u(x,0) = \begin{cases} 2 & x < 2 \\ 3 & x > 2 \end{cases}$$

fan

$$\frac{du}{dt} = 0 \quad \Rightarrow \quad u(x,t) = u(x_0, 0)$$

$$\frac{dx}{dt} = u = u(x_0, 0)$$

$$\Downarrow$$

$$x = t\,u\,(x_0, 0) + x_0$$

For $x_0 < 2$ $\quad x = 2t + x_0 \quad \Rightarrow \quad x - 2t < 2$

For $x_0 > 2$ $\quad x = 3t + x_0 \quad \Rightarrow \quad x - 3t > 2$

$x = t\,u\,(x_0, 0) + x_0$ $\quad$ at discontinuity $x_0 = 2$

we get $x = t\,u + 2$

$$u = \frac{x-2}{t}$$

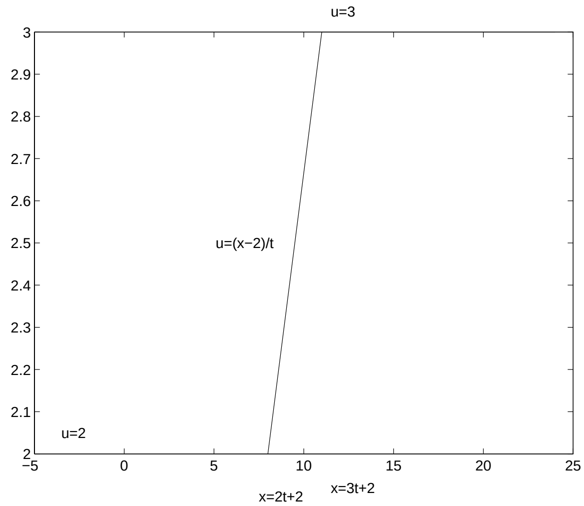

Figure 17: Solution for 5

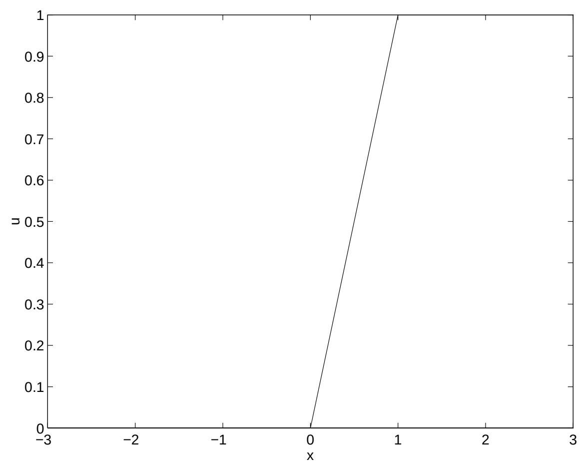

Figure 18: Sketch of initial solution

6.

$$u_t + uu_x = 0$$

$$u(x, 0) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$\frac{du}{dt} = 0 \quad \Rightarrow \quad u(x, t) = u(x_0, 0)$$

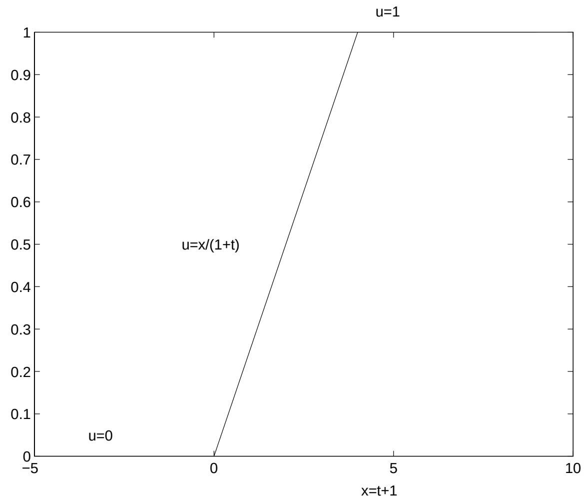

Figure 19: Solution for 6

$$\frac{dx}{dt} = u = u(x_0, 0) \Rightarrow \quad x = t\,u\,(x_0, 0) + x_0$$

For	$x_0 < 0$	$x = t \cdot 0 + x_0$	$\Rightarrow$	$x = x_0$	$u = 0$
	$0 \leq x_0 < 1$	$x = t\,x_0 + x_0$	$\Rightarrow$	$x = x_0\,(1 + t)$	$u = x_0 = \frac{x}{1+t}$
	$1 \leq x_0$	$x = t + x_0$	$\Rightarrow$	$x = t + x_0$	$u = 1$

Basically the interval $[0, 1]$ is stretched in time to $[0, 1 + \mathrm{t}]$.

7.

$$u_t + uu_x = 0$$

$$u(x, 0) = \begin{cases} 2 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad 0 < x < 1 \\ 0 & \text{for} \quad x > 1 \end{cases}$$

First find the shock characteristic for those with speed $u = 2$ and $u = 1$

$$[q] = \frac{1}{2}u^2\Big|_1^2 = \frac{1}{2}(2^2 - 1^2) = \frac{3}{2}$$

$$[u] = 2 - 1 = 1$$

Thus

$$\frac{dx_s}{dt} = \frac{3}{2}$$

and the characteristic through $x = 0$ is then

$$x_s = \frac{3}{2}t$$

Similarly for the shock characteristic for those with speed $u = 1$ and $u = 0$

$$[q] = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}(1^2 - 0^2) = \frac{1}{2}$$

$$[u] = 1 - 0 = 1$$

Thus

$$\frac{dx_s}{dt} = \frac{1}{2}$$

and the characteristic through $x = 1$ is then

$$x_s = \frac{1}{2}t + 1$$

Now these two shock charateristic going to intersect. The point of intersection is found by equating x_s in both, i.e.

$$\frac{1}{2}t + 1 = \frac{3}{2}t$$

The solution is $t = 1$ and $x_s = \dfrac{3}{2}$. Now the speeds are $u = 2$ and $u = 0$

$$[q] = \frac{1}{2}u^2\Big|_0^2 = \frac{1}{2}(2^2 - 0^2) = 2$$

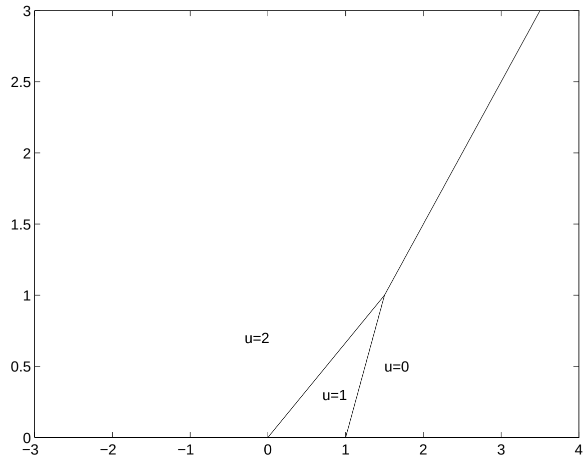

Figure 20: Solution for 7

$$[u] = 2 - 0 = 2$$

Thus

$$\frac{dx_s}{dt} = 1$$

and the characteristic is then

$$x_s = t + C.$$

To find C, we substitute the point of intersection $t = 1$ and $x_s = \frac{3}{2}$. Thus

$$\frac{3}{2} = 1 + C$$

or

$$C = \frac{1}{2}$$

The third shock characteristic is then

$$x_s = t + \frac{1}{2}.$$

The shock characteristics and the solutions in each domain are given in the figure above.

3.3 Second Order Wave Equation

3.3.1 Infinite Domain

Problems

1. Suppose that

$$u(x,t) = F(x - ct).$$

Evaluate

a. $\dfrac{\partial u}{\partial t}(x,0)$

b. $\dfrac{\partial u}{\partial x}(0,t)$

2. The general solution of the one dimensional wave equation

$$u_{tt} - 4u_{xx} = 0$$

is given by

$$u(x,t) = F(x - 2t) + G(x + 2t).$$

Find the solution subject to the initial conditions

$$u(x,0) = \cos x \qquad -\infty < x < \infty,$$

$$u_t(x,0) = 0 \qquad -\infty < x < \infty.$$

3. In section 3.1, we suggest that the wave equation can be written as a system of two first order PDEs. Show how to solve

$$u_{tt} - c^2 u_{xx} = 0$$

using that idea.

1a.

$$u(x,\, t) \,=\, F(x \,-\, ct)$$

Use the chain rule:

$$\frac{\partial\, u}{\partial\, t} \,=\, -\, c\, \frac{dF(x \,-\, ct)}{d(x \,-\, ct)}$$

at $t \,=\, 0$

$$\underline{\frac{\partial\, u}{\partial\, t} \,=\, -\, c\, \frac{dF(x)}{dx}}$$

1b.

$$\frac{\partial\, u}{\partial\, x} \,=\, \frac{dF(x \,-\, ct)}{d(x \,-\, ct)} \,\cdot\, 1$$

at $x \,=\, 0$

$$\frac{\partial\, u}{\partial\, x} \,=\, \frac{dF(-\, ct)}{d(-\, ct)} \,=\, -\, \frac{1}{c}\, \frac{dF(-\, ct)}{dt} \,=\, F'\,(-\, ct)$$

$\uparrow$

differentiation with respect to argument

2. $u(x, t) = F(x - 2t) + G(x + 2t)$

$u(x, 0) = \cos x$

$u_t(x, t) = 0$

$u(x, 0) = F(x) + G(x) = \cos x \qquad (*)$

$u_t(x, t) = -2F'(x - 2t) + 2G'(x + 2t)$

$\Rightarrow u_t(x, 0) = -2F'(x) + 2G'(x) = 0$

Integrate $\Rightarrow -F(x) + G(x) = \text{constant} = k \qquad (\#)$

solve the 2 equations (*) and (#)

$$2G(x) = \cos x + k$$

$$G(x) = \tfrac{1}{2} \cos x + \tfrac{1}{2}k$$

$$2F(x) = \cos x - k$$

$$F(x) = \tfrac{1}{2} \cos x - \tfrac{1}{2}k$$

We need $F(x - 2t) \Rightarrow F(x - 2t) = \frac{1}{2} cos(x - 2t) - \frac{1}{2}k$

$$G(x + 2t) = \tfrac{1}{2} cos(x + 2t) + \tfrac{1}{2}k$$

$\Rightarrow u(x, t) = \frac{1}{2} \cos(x - 2t) + \frac{1}{2} \cos(x + 2t) - \frac{1}{2}k + \frac{1}{2}k$

$$\underline{u(x, t) = \frac{1}{2} \{cos(x - 2t) + \cos(x + 2t)\}}$$

3. The wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

can be written as a system of two first order PDEs

$$\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0$$

and

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = v.$$

Solving the first for v, by rewriting it as a system of ODEs

$$\frac{dv}{dt} = 0$$

$$\frac{dx}{dt} = -c$$

The characteristic equation is solved

$$x = -ct + x_0$$

and then

$$v(x,t) = v(x_0, 0) = V(x + ct)$$

where V is the initial solution for v. Now use this solution in the second PDE rewritten as a system of ODEs

$$\frac{du}{dt} = V(x + ct)$$

$$\frac{dx}{dt} = c$$

The characteristic equation is solved

$$x = ct + x_0$$

and then

$$\frac{du}{dt} = V(x + ct) = V(x_0 + 2ct)$$

Integrating

$$u(x_0, t) = \int_0^t V(x_0 + 2c\tau)d\tau + K(x_0)$$

Change variables

$$z = x_0 + 2c\tau$$

then

$$dz = 2cd\tau$$

The limits of integration become x_0 and $x_0 + 2ct$. Thus the solution

$$u(x_0, t) \;=\; \int_{x_0}^{x_0+2ct} \frac{1}{2c} V(z)dz \;+\; K(x_0)$$

But $x_0 \;=\; x - ct$

$$u(x, t) \;=\; \int_{x-ct}^{x+ct} \frac{1}{2c} V(z)dz \;+\; K(x - ct)$$

Now break the integral using the point zero.

$$u(x, t) \;=\; K(x - ct) \;-\; \int_{0}^{x-ct} \frac{1}{2c} V(z)dz \;+\; \int_{0}^{x+ct} \frac{1}{2c} V(z)dz$$

The first two terms give a function of $x - ct$ and the last term is a function of $x + ct$, exactly as we expect from D'Alembert solution.

3.3.2 Semi-infinite String

3.3.3 Semi-infinite String with a Free End

Problems

1. Solve by the method of characteristics

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad x > 0$$

subject to

$$u(x,0) = 0,$$
$$\frac{\partial u}{\partial t}(x,0) = 0,$$
$$u(0,t) = h(t).$$

2. Solve

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad x < 0$$

subject to

$$u(x,0) = \sin x, \qquad x < 0$$
$$\frac{\partial u}{\partial t}(x,0) = 0, \qquad x < 0$$
$$u(0,t) = e^{-t}, \qquad t > 0.$$

3. a. Solve

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < \infty$$

subject to

$$u(x,0) = \begin{cases} 0 & 0 < x < 2 \\ 1 & 2 < x < 3 \\ 0 & 3 < x \end{cases}$$
$$\frac{\partial u}{\partial t}(x,0) = 0,$$
$$\frac{\partial u}{\partial x}(0,t) = 0.$$

b. Suppose u is continuous at $x = t = 0$, sketch the solution at various times.

4. Solve

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad x > 0, \qquad t > 0$$

subject to

$$u(x,0) = 0,$$

$$\frac{\partial u}{\partial t}(x,0) = 0,$$
$$\frac{\partial u}{\partial x}(0,t) = h(t).$$

5. Give the domain of influence in the case of semi-infinite string.

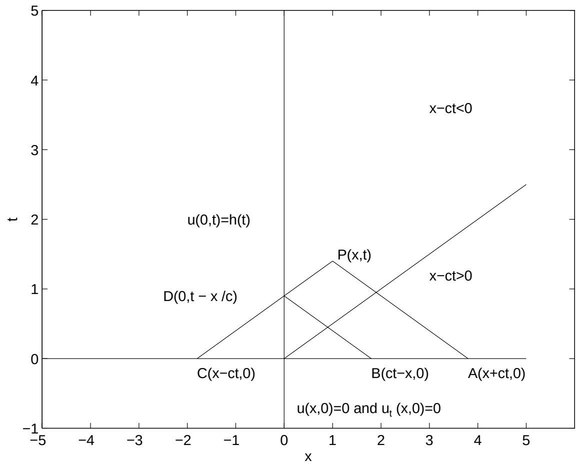

Figure 21: Domain for problem 1

1. $u_{tt} - c^2 u_{xx} = 0$

$u(x, 0) = 0$

$u_t(x, 0) = 0$

$u(0, t) = h(t)$

Solution $\quad u(x, t) = F(x - ct) + G(x + ct)$

(*)

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(\tau)\, d\tau = 0$$

(#)

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(\tau)\, d\tau = 0$$

since both $f(x)$, $g(x)$ are zero.

Thus for $x - ct > 0$ the solution is zero

(No influence of boundary at $x = 0$)

For $x - ct < 0$ $\quad u(0, t) = h(t)$

$$\Downarrow$$

$$F(-ct) + G(ct) = h(t)$$

$u(x, t) = F(x - ct) + G(x + ct)$

but argument of F is negative and thus we cannot use (*), instead

$F(-\,ct) \,=\, h(t) \,-\, G(ct)$

or $F(z) \,=\, h(\frac{-z}{c}) \,-\, G(-z)$ for $z \,<\, 0$

$$F(x \,-\, ct) \,=\, h(-\tfrac{x\,-\,ct}{c}) \,-\, G(-(x \,-\, ct))$$

$$= h(t \,-\, \tfrac{x}{c}) \,-\, G(ct \,-\, x)$$

$$u(x,\, t) \,=\, F(x \,-\, ct) \,+\, G(x \,+\, ct)$$

$$= h(t \,-\, \tfrac{x}{c}) \,-\, \underbrace{G(ct \,-\, x)}_{\text{zero}} + \underbrace{G(x \,+\, ct)}_{\text{zero}}$$

since the arguments are positive and (#) is valid

$$\Rightarrow \quad u(x,\, t) \,=\, h(t \,-\, \tfrac{x}{c}) \quad \text{for} \quad 0 \,<\, x \,<\, ct$$

$$u(x,\, t) \,=\, 0 \qquad\qquad \text{for} \quad x \,-\, ct \,>\, 0$$

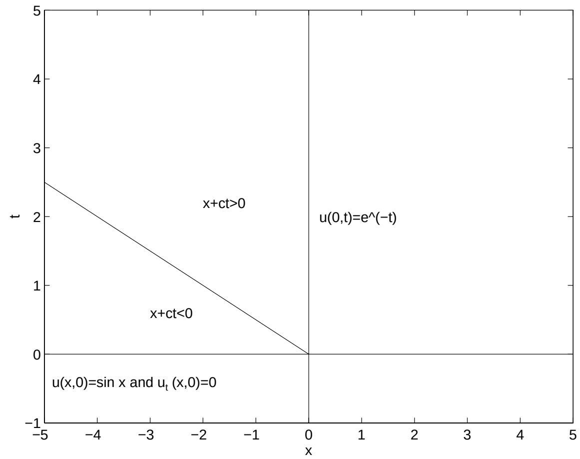

Figure 22: Domain for problem 2

2. $u_{tt} - c^2 u_{xx} = 0 \qquad x < 0$

$u(x, 0) = \sin x \qquad x < 0$

$u_t(x, 0) = 0 \qquad x < 0$

$u(0, t) = e^{-t} \qquad t > 0$

$u(x, t) = F(x - ct) + G(x + ct)$

$F(x) = \frac{1}{2} \sin x$

since $f = \sin x,\ g = 0$

$G(x) = \frac{1}{2} \sin x$

From boundary condition

$u(0, t) = F(-ct) + G(ct) = e^{-t}$

If $x + ct < 0$ no influence of boundary at $x = 0$

$$\Rightarrow \quad u(x, t) = \frac{1}{2} \sin(x - ct) + \frac{1}{2} \sin(x + ct)$$

$$= \sin x \cos ct$$

$\vdots$

after some trigonometric manipulation

If $x + ct > 0$ then the argument of G is positive and thus

$$G(ct) = e^{-t} - F(-ct)$$

or $\quad G(z) = e^{-z/c} - F(-z)$

$$\Rightarrow \quad G(x + ct) = e^{-\frac{x + ct}{c}} - F(-(x + ct))$$

Therefore:

$$u(x,\,t)\;=\;F\,(x\;-\;ct)\;+\;G(x\;+\;ct)$$

$$=\;F\,(x\;-\;ct)\;+\;e^{-\frac{x\,+\,ct}{c}}\;-\;F\,(-\,x\;-\;ct)$$

$$=\;\tfrac{1}{2}\;\sin\,(x\;-\;ct)\;+\;e^{-\frac{x\,+\,ct}{c}}\;-\;\tfrac{1}{2}\;\underbrace{\sin\,(-\,x\;-\;ct)}_{-\;\sin\,(x\,+\,ct)}$$

$$=\;\underbrace{\frac{1}{2}\;\sin\,(x\;-\;ct)\;-\;\frac{1}{2}\;\sin\,(x\;+\;ct)}_{\cos\;ct\;\sin\;x}\;+\,e^{-\frac{x\,+\,ct}{c}}$$

$$\Rightarrow u(x,\,t)\;=\begin{cases}\sin x\cos ct & x+ct<0\\[2ex] \sin x\cos ct+e^{-\frac{x\,+\,ct}{c}} & x+ct>0\end{cases}$$

3a. $u_{tt} = c^2 u_{xx} \qquad 0 < x < \infty$

$u_x(0, t) = 0$

$$u(x, 0) = \begin{cases} 0 & 0 < x < 2 \\ 1 & 2 < x < 3 \\ 0 & x > 3 \end{cases}$$

$u_t(x, 0) = 0$

$u(x, t) = F(x - ct) + G(x + ct)$

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(\xi)\, d\xi = \frac{1}{2} f(x) \qquad g \overset{since}{\equiv} 0 \qquad x > 0$$

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(\xi)\, d\xi = \frac{1}{2} f(x) \qquad g \equiv 0 \qquad x > 0$$

$$u(x, t) = \frac{f(x + ct) + f(x - ct)}{2}, \qquad x > ct$$

$u_x(0, t) = F'(-ct) + G'(ct) = 0$

$\Rightarrow \quad F'(-ct) = -G'(ct)$

$F'(-z) = -G'(z)$

Integrate

$$-F(-z) = -G(z) + K$$

$$\boxed{F(-z) = G(z) - K}$$

$\Rightarrow \quad F(x - ct) = -G(ct - x) - K \qquad \underline{x - ct < 0}$

$= \frac{1}{2} f(ct - x) - K \qquad \underline{x - ct < 0}$

$\Rightarrow \quad u(x, t) = \frac{1}{2} f(x + ct) + \frac{1}{2} f(ct - x) - K$

To find K we look at $x = 0, t = 0 \qquad u(0, 0) = 0$ from initial condition

but $u(0, 0) = \frac{1}{2} f(0) + \frac{1}{2} f(0) - K = \underbrace{f(0)}_{=0} - K$ from above

$\Rightarrow \quad K = 0$

$$\Rightarrow \quad u(x, t) = \frac{f(x + ct) + f(ct - x)}{2}$$

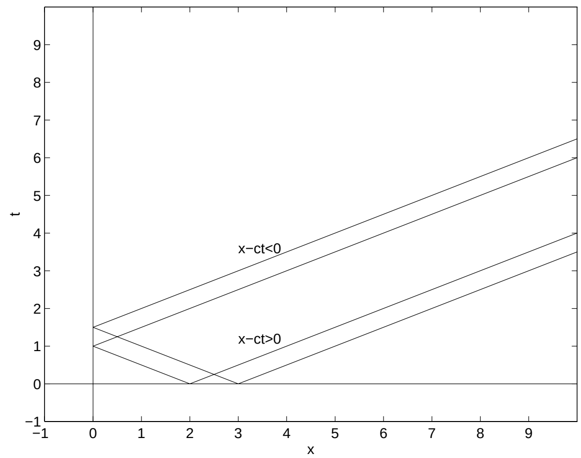

Figure 23: Domain of influence for problem 3

3b.

$$u(x,\, t) \;=\; \begin{cases} \dfrac{f(x \,+\, ct) \,+\, f(x \,-\, ct)}{2} & x > ct \\[2ex] \dfrac{f(x \,+\, ct) \,+\, f(ct \,-\, x)}{2} & x < ct \end{cases}$$

where

$$u(x,\, t) \;=\; \begin{cases} 1 & \text{Region I} \\[2ex] \dfrac{1}{2} & \text{Region II} \\[2ex] 0 & \text{otherwise} \end{cases}$$

In order to find the regions I and II mentioned above, we use the idea of domain of influence. Sketch both characteristics from the end points of the interval (2,3) and remember that when the characteristic curve (line in this case) reaches the t axis, it will be reflected.

As can be seen in the figure, the only region where the solution is 1 is the two triangular regions. Within the three strips (not including the above mentioned triangles), the solution is $\dfrac{1}{2}$ and for the rest, the solution is zero.

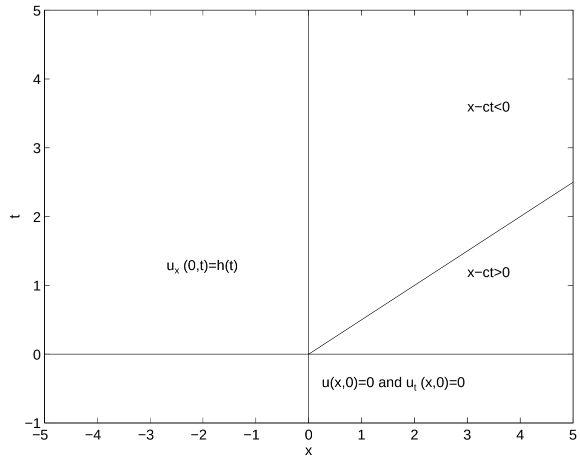

Figure 24: Domain for problem 4

4. $u_{tt} - c^2 u_{xx} = 0$

general solution

$u(x,\, t) = F(x - ct) + G(x + ct)$
For $\quad x - ct > 0 \qquad u(x,\, t) = 0 \qquad$ since $\qquad u = u_t = 0$ on the boundary.

For $x - ct < 0$ we get the influence of the boundary condition

$$u_x(0,\, t) = h(t)$$

Differentiate the general solution:

$u_x(x,\, t) = F'(x - ct) \cdot 1 + G'(x + ct) \cdot 1 = \frac{dF(x-ct)}{d(x-ct)} + \frac{dG(x+ct)}{d(x+ct)}$
$\vdots$
chain rule
prime means derivative with respect to argument

As $x = 0$:

$h(t) = u_x(0,\, t) = \frac{dF(-ct)}{d(-ct)} + \frac{dG(ct)}{d(ct)} = -\frac{1}{c}\frac{dF(-ct)}{dt} + \frac{1}{c}\frac{dG(ct)}{dt}$

Integrate

$$-\frac{1}{c}\,F(-\,ct)\;+\;\frac{1}{c}\,\underbrace{G(ct)}_{=\,0}+\frac{1}{c}\,\underbrace{F(0)}_{=\,0}-\frac{1}{c}\,\underbrace{G(0)}_{=\,0}\;=\;\int_0^t\,h(\tau)\,d\tau$$

since $f\;=\;g\;=\;0$

$$F(-\,ct)\;=\;-c\int_0^t\,h(\tau)\,d\tau$$

$$F(z)\;=\;-c\int_0^{-z/c}\,h(\tau)\,d\tau$$

$$\Rightarrow\quad F(x\;-\;ct)\;=\;-c\int_0^{-\frac{x\,-\,ct}{c}}\,h(\tau)\,d\tau$$

$$\Rightarrow u(x,\,t)\;=\begin{cases}0 & x-ct>0\\[2ex] -\,c\displaystyle\int_0^{-\frac{x\,-\,ct}{c}}\,h(\tau)\,d\tau & x-ct<0\end{cases}$$

5. For the infinite string the domain of influence is a wedge with vertex at the point of interest $(x, 0)$. For the semi infinite string, the left characteristic is reflected by the vertical t axis and one obtains a strip, with one along a characteristic $(x + ct = C)$ reaching the t axis and the other two sides are from the other family of characteristcs $(x - ct = K)$.

CHAPTER 4

4 Separation of Variables-Homogeneous Equations

4.1 Parabolic equation in one dimension

4.2 Other Homogeneous Boundary Conditions

Problems

1. Consider the differential equation

$$X''(x) + \lambda X(x) = 0$$

Determine the eigenvalues λ (assumed real) subject to

a. $X(0) = X(\pi) = 0$

b. $X'(0) = X'(L) = 0$

c. $X(0) = X'(L) = 0$

d. $X'(0) = X(L) = 0$

e. $X(0) = 0$ and $X'(L) + X(L) = 0$

Analyze the cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

1. a.

$$X'' + \lambda X = 0$$
$$X(0) = 0$$
$$X(\pi) = 0$$

Try e^{rx}. As we know from ODEs, this leads to the characteristic equation for r

$$r^2 + \lambda = 0$$

Or

$$r = \pm\sqrt{-\lambda}$$

We now consider three cases depending on the sign of λ

Case 1: $\lambda < 0$

In this case r is the square root of a positive number and thus we have two real roots. In this case the solution is a linear combination of two real exponentials

$$X = A_1 e^{\sqrt{-\lambda}x} + B_1 e^{-\sqrt{-\lambda}x}$$

It is well known that the solution can also be written as a combination of hyperbolic sine and cosine, i.e.

$$X = A_2 \cosh\sqrt{-\lambda}x + B_2 \sinh\sqrt{-\lambda}x$$

The other two forms are may be less known, but easily proven. The solution can be written as a shifted hyperbolic cosine (sine). The proof is straight forward by using the formula for $\cosh(a+b)$ $(\sinh(a+b))$

$$X = A_3 \cosh\left(\sqrt{-\lambda}x + B_3\right)$$

Or

$$X = A_4 \sinh\left(\sqrt{-\lambda}x + B_4\right)$$

Which form to use, depends on the boundary conditions. Recall that the hypebolic sine vanishes ONLY at $x = 0$ and the hyperbolic cosine is always positive. If we use the last form of the general solution then we immediately find that $B_4 = 0$ is a result of the first boundary condition and clearly to satisfy the second boundary condition we must have $A_4 = 0$ (recall $\sinh x = 0$ only for $x = 0$ and the second boundary condition reads $A_4 \sinh\sqrt{-\lambda}\pi = 0$, thus the coefficient A_4 must vanish).

Any other form will yields the same trivial solution, may be with more work!!!

Case 2: $\lambda = 0$

In this case we have a double root $r = 0$ and as we know from ODEs the solution is

$$X = Ax + B$$

The boundary condition at zero yields

$$X(0) = B = 0$$

and the second condition

$$X(\pi) = A\pi = 0$$

This implies that $A = 0$ and therefore we again have a trivial solution.

Case 3: $\lambda > 0$
In this case the two roots are imaginary

$$r = \pm i\sqrt{\lambda}$$

Thus the solution is a combination of sine and cosine

$$X = A_1 \cos\sqrt{\lambda}x + B_1 \sin\sqrt{\lambda}x$$

Substitute the boundary condition at zero

$$X(0) = A_1$$

Thus $A_1 = 0$ and the solution is

$$X = B_1 \sin\sqrt{\lambda}x$$

Now use the condition at π

$$X(\pi) = B_1 \sin\sqrt{\lambda}\pi = 0$$

If we take $B_1 = 0$, we get a trivial solution, but we have another choice, namely

$$\sin\sqrt{\lambda}\pi = 0$$

This implies that the argument of the sine function is a multiple of π

$$\sqrt{\lambda_n}\pi = n\pi \qquad n = 1, 2, \ldots$$

Notice that since $\lambda > 0$ we must have $n > 0$. Thus

$$\sqrt{\lambda_n} = n \qquad n = 1, 2, \ldots$$

Or

$$\lambda_n = n^2 \qquad n = 1, 2, \ldots$$

The solution is then depending on n, and obtained by substituting for λ_n

$$X_n(x) = \sin nx$$

Note that we ignored the constant B_1 since the eigenfunctions are determined up to a multiplicative constant. (We will see later that the constant will be incorporated with that of the linear combination used to get the solution for the PDE)

1. b.

$$X'' + \lambda X = 0$$
$$X'(0) = 0$$
$$X'(L) = 0$$

Try e^{rx}. As we know from ODEs, this leads to the characteristic equation for r

$$r^2 + \lambda = 0$$

Or

$$r = \pm\sqrt{-\lambda}$$

We now consider three cases depending on the sign of λ

Case 1: $\lambda < 0$

In this case r is the square root of a positive number and thus we have two real roots. In this case the solution is a linear combination of two real exponentials

$$X = A_1 e^{\sqrt{-\lambda}x} + B_1 e^{-\sqrt{-\lambda}x}$$

It is well known that the solution can also be written as a combination of hyperbolic sine and cosine, i.e.

$$X = A_2 \cosh \sqrt{-\lambda}x + B_2 \sinh \sqrt{-\lambda}x$$

The other two forms are may be less known, but easily proven. The solution can be written as a shifted hyperbolic cosine (sine). The proof is straight forward by using the formula for $\cosh(a + b)$ $(\sinh(a + b))$

$$X = A_3 \cosh \left(\sqrt{-\lambda}x + B_3\right)$$

Or

$$X = A_4 \sinh \left(\sqrt{-\lambda}x + B_4\right)$$

Which form to use, depends on the boundary conditions. Recall that the hypebolic sine vanishes ONLY at $x = 0$ and the hyperbolic cosine is always positive. If we use the following form of the general solution

$$X = A_3 \cosh \left(\sqrt{-\lambda}x + B_3\right)$$

then the derivative X' wil be

$$X' = \sqrt{-\lambda} A_3 \sinh \left(\sqrt{-\lambda}x + B_3\right)$$

The first boundary condition $X'(0) =$ yields $B_3 = 0$ and clearly to satisfy the second boundary condition we must have $A_3 = 0$ (recall $\sinh x = 0$ only for $x = 0$ and the second boundary condition reads $\sqrt{-\lambda} A_3 \sinh \sqrt{-\lambda} L = 0$, thus the coefficient A_3 must vanish).

Any other form will yields the same trivial solution, may be with more work!!!

Case 2: $\lambda = 0$

In this case we have a double root $r = 0$ and as we know from ODEs the solution is

$$X = Ax + B$$

The boundary condition at zero yields

$$X'(0) = A = 0$$

and the second condition

$$X'(L) = A = 0$$

This implies that $A = 0$ and therefore we have no restriction on B. Thus in this case the solution is a constant and we take

$$X(x) = 1$$

Case 3: $\lambda > 0$

In this case the two roots are imaginary

$$r = \pm i\sqrt{\lambda}$$

Thus the solution is a combination of sine and cosine

$$X = A_1 \cos\sqrt{\lambda}x + B_1 \sin\sqrt{\lambda}x$$

Differentiate

$$X' = -\sqrt{\lambda}A_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}B_1 \cos\sqrt{\lambda}x$$

Substitute the boundary condition at zero

$$X'(0) = \sqrt{\lambda}B_1$$

Thus $B_1 = 0$ and the solution is

$$X = A_1 \cos\sqrt{\lambda}x$$

Now use the condition at L

$$X'(L) = -\sqrt{\lambda}A_1 \sin\sqrt{\lambda}L = 0$$

If we take $A_1 = 0$, we get a trivial solution, but we have another choice, namely

$$\sin\sqrt{\lambda}L = 0$$

This implies that the argument of the sine function is a multiple of π

$$\sqrt{\lambda_n}L = n\pi \qquad n = 1, 2, \ldots$$

Notice that since $\lambda > 0$ we must have $n > 0$. Thus

$$\sqrt{\lambda_n} = \frac{n\pi}{L} \qquad n = 1, 2, \ldots$$

Or

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \ldots$$

The solution is then depending on n, and obtained by substituting λ_n

$$X_n(x) = \cos\frac{n\pi}{L}x$$

1. c.

$$X'' + \lambda X = 0$$

$$X(0) = 0$$

$$X'(L) = 0$$

Try e^{rx}. As we know from ODEs, this leads to the characteristic equation for r

$$r^2 + \lambda = 0$$

Or

$$r = \pm\sqrt{-\lambda}$$

We now consider three cases depending on the sign of λ

Case 1: $\lambda < 0$

In this case r is the square root of a positive number and thus we have two real roots. In this case the solution is a linear combination of two real exponentials

$$X = A_1 e^{\sqrt{-\lambda}x} + B_1 e^{-\sqrt{-\lambda}x}$$

It is well known that the solution can also be written as a combination of hyperbolic sine and cosine, i.e.

$$X = A_2 \cosh\sqrt{-\lambda}x + B_2 \sinh\sqrt{-\lambda}x$$

The other two forms are may be less known, but easily proven. The solution can be written as a shifted hyperbolic cosine (sine). The proof is straight forward by using the formula for $\cosh(a+b)$ $(\sinh(a+b))$

$$X = A_3 \cosh\left(\sqrt{-\lambda}x + B_3\right)$$

Or

$$X = A_4 \sinh\left(\sqrt{-\lambda}x + B_4\right)$$

Which form to use, depends on the boundary conditions. Recall that the hypebolic sine vanishes ONLY at $x = 0$ and the hyperbolic cosine is always positive. If we use the following form of the general solution

$$X = A_4 \sinh\left(\sqrt{-\lambda}x + B_4\right)$$

then the derivative X' wil be

$$X' = \sqrt{-\lambda}A_4 \cosh\left(\sqrt{-\lambda}x + B_4\right)$$

The first boundary condition $X(0) =$ yields $B_4 = 0$ and clearly to satisfy the second boundary condition we must have $A_4 = 0$ (recall $\cosh x$ is never zero thus the coefficient A_4 must vanish).

Any other form will yields the same trivial solution, may be with more work!!!

Case 2: $\lambda = 0$

In this case we have a double root $r = 0$ and as we know from ODEs the solution is

$$X = Ax + B$$

The boundary condition at zero yields

$$X(0) = B = 0$$

and the second condition

$$X'(L) = A = 0$$

This implies that $B = A = 0$ and therefore we have again a trivial solution.

Case 3: $\lambda > 0$

In this case the two roots are imaginary

$$r = \pm i\sqrt{\lambda}$$

Thus the solution is a combination of sine and cosine

$$X = A_1 \cos \sqrt{\lambda}x + B_1 \sin \sqrt{\lambda}x$$

Differentiate

$$X' = -\sqrt{\lambda}A_1 \sin \sqrt{\lambda}x + \sqrt{\lambda}B_1 \cos \sqrt{\lambda}x$$

Substitute the boundary condition at zero

$$X(0) = A_1$$

Thus $A_1 = 0$ and the solution is

$$X = B_1 \sin \sqrt{\lambda}x$$

Now use the condition at L

$$X'(L) = \sqrt{\lambda}B_1 \cos \sqrt{\lambda}L = 0$$

If we take $B_1 = 0$, we get a trivial solution, but we have another choice, namely

$$\cos \sqrt{\lambda}\mathrm{L} = 0$$

This implies that the argument of the cosine function is a multiple of π plus $\pi/2$

$$\sqrt{\lambda_n}L = \left(n + \frac{1}{2}\right)\pi \qquad n = 0, 1, 2, \ldots$$

Notice that since $\lambda > 0$ we must have $n \geq 0$. Thus

$$\sqrt{\lambda_n} = \frac{\left(n + \frac{1}{2}\right)\pi}{L} \qquad n = 0, 1, 2, \ldots$$

Or

$$\lambda_n = \left(\frac{\left(n + \frac{1}{2}\right)\pi}{L}\right)^2 \qquad n = 0, 1, 2, \ldots$$

The solution is then depending on n, and obtained by substituting λ_n

$$X_n(x) = \sin \frac{\left(n + \frac{1}{2}\right)\pi}{L} x$$

1. d.

$$X'' + \lambda X = 0$$
$$X'(0) = 0$$
$$X(L) = 0$$

Try e^{rx}. As we know from ODEs, this leads to the characteristic equation for r

$$r^2 + \lambda = 0$$

Or

$$r = \pm\sqrt{-\lambda}$$

We now consider three cases depending on the sign of λ

Case 1: $\lambda < 0$

In this case r is the square root of a positive number and thus we have two real roots. In this case the solution is a linear combination of two real exponentials

$$X = A_1 e^{\sqrt{-\lambda}x} + B_1 e^{-\sqrt{-\lambda}x}$$

It is well known that the solution can also be written as a combination of hyperbolic sine and cosine, i.e.

$$X = A_2 \cosh\sqrt{-\lambda}x + B_2 \sinh\sqrt{-\lambda}x$$

The other two forms are may be less known, but easily proven. The solution can be written as a shifted hyperbolic cosine (sine). The proof is straight forward by using the formula for $\cosh(a+b)$ $(\sinh(a+b))$

$$X = A_3 \cosh\left(\sqrt{-\lambda}x + B_3\right)$$

Or

$$X = A_4 \sinh\left(\sqrt{-\lambda}x + B_4\right)$$

Which form to use, depends on the boundary conditions. Recall that the hypebolic sine vanishes ONLY at $x = 0$ and the hyperbolic cosine is always positive. If we use the following form of the general solution

$$X = A_3 \cosh\left(\sqrt{-\lambda}x + B_3\right)$$

then the derivative X' wil be

$$X' = \sqrt{-\lambda}A_3 \sinh\left(\sqrt{-\lambda}x + B_3\right)$$

The first boundary condition $X'(0) =$ yields $B_3 = 0$ and clearly to satisfy the second boundary condition we must have $A_3 = 0$ (recall $\cosh x$ is never zero thus the coefficient A_3 must vanish).

Any other form will yields the same trivial solution, may be with more work!!!

Case 2: $\lambda = 0$

In this case we have a double root $r = 0$ and as we know from ODEs the solution is

$$X = Ax + B$$

The boundary condition at zero yields

$$X'(0) = A = 0$$

and the second condition

$$X(L) = B = 0$$

This implies that $B = A = 0$ and therefore we have again a trivial solution.

Case 3: $\lambda > 0$

In this case the two roots are imaginary

$$r = \pm i\sqrt{\lambda}$$

Thus the solution is a combination of sine and cosine

$$X = A_1 \cos \sqrt{\lambda}x + B_1 \sin \sqrt{\lambda}x$$

Differentiate

$$X' = -\sqrt{\lambda}A_1 \sin \sqrt{\lambda}x + \sqrt{\lambda}B_1 \cos \sqrt{\lambda}x$$

Substitute the boundary condition at zero

$$X'(0) = \sqrt{\lambda}B_1$$

Thus $B_1 = 0$ and the solution is

$$X = A_1 \cos \sqrt{\lambda}x$$

Now use the condition at L

$$X(L) = A_1 \cos \sqrt{\lambda}L = 0$$

If we take $A_1 = 0$, we get a trivial solution, but we have another choice, namely

$$\cos \sqrt{\lambda}\text{Ł} = 0$$

This implies that the argument of the cosine function is a multiple of π plus $\pi/2$

$$\sqrt{\lambda_n}L = \left(n + \frac{1}{2}\right)\pi \qquad n = 0, 1, 2, \ldots$$

Notice that since $\lambda > 0$ we must have $n \geq 0$. Thus

$$\sqrt{\lambda_n} = \frac{\left(n + \frac{1}{2}\right)\pi}{L} \qquad n = 0, 1, 2, \ldots$$

Or

$$\lambda_n = \left(\frac{\left(n + \frac{1}{2}\right)\pi}{L}\right)^2 \qquad n = 0, 1, 2, \ldots$$

The solution is then depending on n, and obtained by substituting λ_n

$$X_n(x) = \cos \frac{\left(n + \frac{1}{2}\right)\pi}{L}x$$

1. e.

$$X'' + \lambda X = 0$$

$$X(0) = 0$$

$$X'(L) + X(L) = 0$$

Try e^{rx}. As we know from ODEs, this leads to the characteristic equation for r

$$r^2 + \lambda = 0$$

Or

$$r = \pm\sqrt{-\lambda}$$

We now consider three cases depending on the sign of λ

Case 1: $\lambda < 0$

In this case r is the square root of a positive number and thus we have two real roots. In this case the solution is a linear combination of two real exponentials

$$X = A_1 e^{\sqrt{-\lambda}x} + B_1 e^{-\sqrt{-\lambda}x}$$

It is well known that the solution can also be written as a combination of hyperbolic sine and cosine, i.e.

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$$X = A_3 \cosh\left(\sqrt{-\lambda}x + B_3\right)$$

Or

$$X = A_4 \sinh\left(\sqrt{-\lambda}x + B_4\right)$$

Which form to use, depends on the boundary conditions. Recall that the hypebolic sine vanishes ONLY at $x = 0$ and the hyperbolic cosine is always positive. If we use the following form of the general solution

$$X = A_4 \sinh\left(\sqrt{-\lambda}x + B_4\right)$$

then the derivative X' wil be

$$X' = \sqrt{-\lambda}A_4 \cosh\left(\sqrt{-\lambda}x + B_4\right)$$

The first boundary condition $X(0) = 0$ yields $B_4 = 0$ and clearly to satisfy the second boundary condition

$$\sqrt{-\lambda}A_4 \cosh\sqrt{-\lambda}L = 0$$

we must have $A_4 = 0$ (recall $\cosh x$ is never zero thus the coefficient A_4 must vanish).

Any other form will yields the same trivial solution, may be with more work!!!

Case 2: $\lambda = 0$
In this case we have a double root $r = 0$ and as we know from ODEs the solution is

$$X = Ax + B$$

The boundary condition at zero yields

$$X(0) = B = 0$$

and the second condition

$$X'(L) + X(L) = A + AL = 0$$

Or

$$A(1 + L) = 0$$

This implies that $B = A = 0$ and therefore we have again a trivial solution.

Case 3: $\lambda > 0$
In this case the two roots are imaginary

$$r = \pm i\sqrt{\lambda}$$

Thus the solution is a combination of sine and cosine

$$X = A_1 \cos\sqrt{\lambda}x + B_1 \sin\sqrt{\lambda}x$$

Differentiate

$$X' = -\sqrt{\lambda}A_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}B_1 \cos\sqrt{\lambda}x$$

Substitute the boundary condition at zero

$$X(0) = A_1$$

Thus $A_1 = 0$ and the solution is

$$X = B_1 \sin\sqrt{\lambda}x$$

Now use the condition at L

$$X'(L) + X(L) = \sqrt{\lambda}B_1 \cos\sqrt{\lambda}L + B_1 \sin\sqrt{\lambda}L = 0$$

If we take $B_1 = 0$, we get a trivial solution, but we have another choice, namely

$$\sqrt{\lambda}\cos\sqrt{\lambda}L + \sin\sqrt{\lambda}L = 0$$

If $\cos\sqrt{\lambda}L = 0$ then we are left with $\sin\sqrt{\lambda}L = 0$ which is not possible (the cosine and sine functions do not vanish at the same points).

Thus $\cos\sqrt{\lambda}L \neq 0$ and upon dividing by it we get

$$-\sqrt{\lambda} = \tan\sqrt{\lambda}L$$

This can be solved graphically or numerically (see figure). The points of intersection are values of $\sqrt{\lambda_n}$. The solution is then depending on n, and obtained by substituting λ_n

$$X_n(x) = \sin\sqrt{\lambda_n}x$$

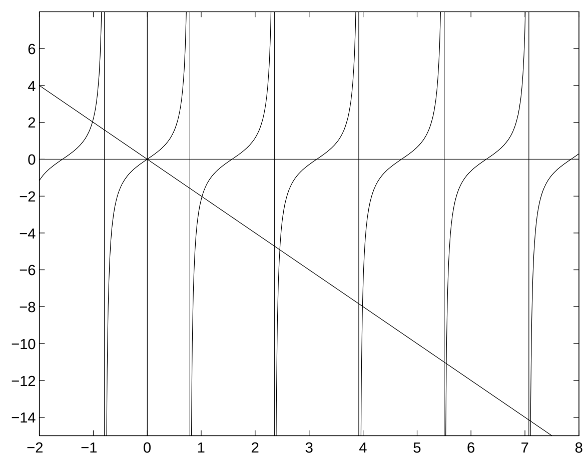


Figure 25: Graphical solution of the eigenvalue problem

CHAPTER 5

5 Fourier Series

5.1 Introduction

5.2 Orthogonality

5.3 Computation of Coefficients

Problems

1. For the following functions, sketch the Fourier series of $f(x)$ on the interval $[-L, L]$. Compare $f(x)$ to its Fourier series

a. $f(x) = 1$

b. $f(x) = x^2$

c. $f(x) = e^x$

d.

$$f(x) = \begin{cases} \frac{1}{2}x & x < 0 \\ 3x & x > 0 \end{cases}$$

e.

$$f(x) = \begin{cases} 0 & x < \frac{L}{2} \\ \\ x^2 & x > \frac{L}{2} \end{cases}$$

2. Sketch the Fourier series of $f(x)$ on the interval $[-L, L]$ and evaluate the Fourier coefficients for each

a. $f(x) = x$

b. $f(x) = \sin \frac{\pi}{L} x$

c.

$$f(x) = \begin{cases} 1 & |x| < \frac{L}{2} \\ \\ 0 & |x| > \frac{L}{2} \end{cases}$$

3. Show that the Fourier series operation is linear, i.e. the Fourier series of $\alpha f(x) + \beta g(x)$ is the sum of the Fourier series of $f(x)$ and $g(x)$ multiplied by the corresponding constant.

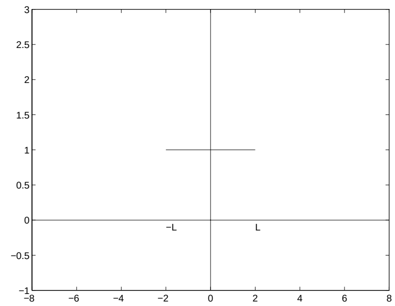


Figure 26: Graph of $f(x) = 1$

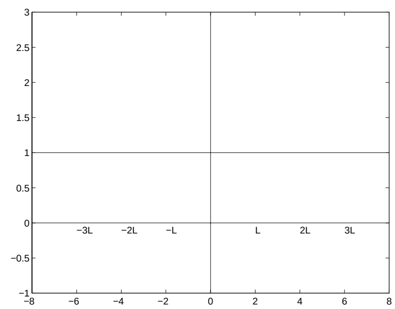


Figure 27: Graph of its periodic extension

1. a. $f(x) = 1$

Since the periodic extension of $f(x)$ is continuous, the Fourier series is identical to (the periodic extension of) $f(x)$ everywhere.

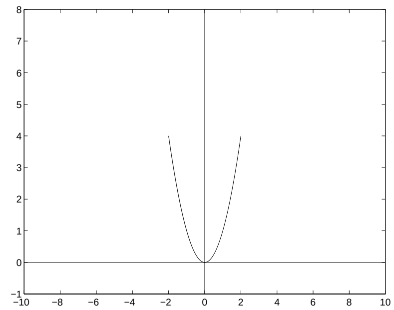


Figure 28: Graph of $f(x) = x^2$

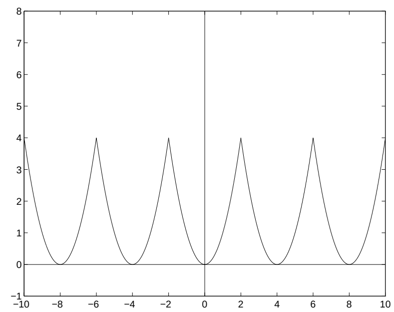


Figure 29: Graph of its periodic extension

1. b. $f(x) = x^2$

Since the periodic extension of $f(x)$ is continuous, the Fourier series is identical to (the periodic extension of) $f(x)$ everywhere.

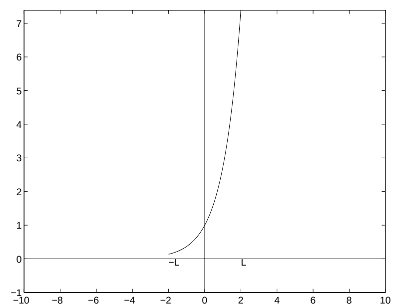


Figure 30: Graph of $f(x) = e^x$

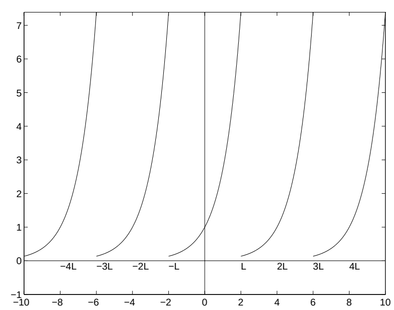


Figure 31: Graph of its periodic extension

1. c. $f(x) = e^x$

Since the periodic extension of $f(x)$ is discontinuous, the Fourier series is identical to (the periodic extension of) $f(x)$ everywhere except for the points of discontinuities. At $x = \pm L$ (and similar points in each period), we have the average value, i.e.

$$\frac{e^L + e^{-L}}{2} = \cosh L$$

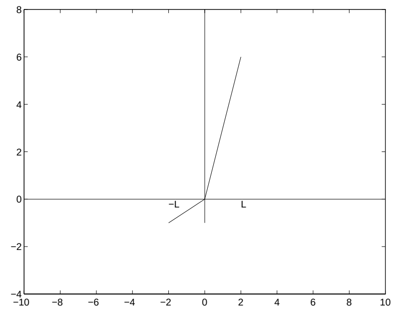


Figure 32: Graph of $f(x)$

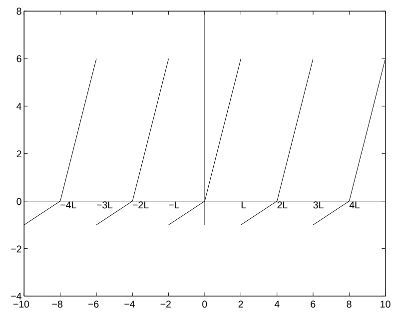


Figure 33: Graph of its periodic extension

1. d.

Since the periodic extension of $f(x)$ is discontinuous, the Fourier series is identical to (the periodic extension of) $f(x)$ everywhere except at the points of discontinuities. At those points $x = \pm L$ (and similar points in each period), we have

$$\frac{3L + (-\frac{1}{2}L)}{2} = \frac{5}{4}L$$

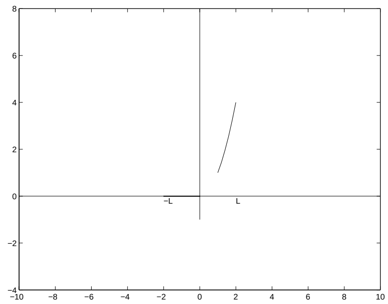


Figure 34: Graph of $f(x)$

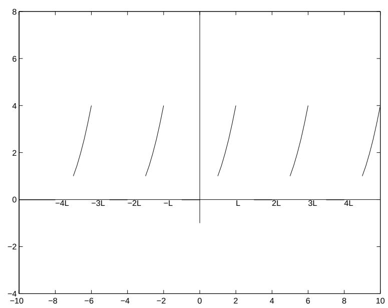


Figure 35: Graph of its periodic extension

1. e.

Since the periodic extension of $f(x)$ is discontinuous, the Fourier series is identical to (the periodic extension of) $f(x)$ everywhere except at the points of discontinuities. At those points $x = \pm L$ (and similar points in each period), we have

$$\frac{L^2 + 0}{2} = \frac{1}{2}L^2$$

At the point $L/2$ and similar ones in each period we have

$$\frac{0 + \frac{1}{4}L^2}{2} = \frac{1}{8}L^2$$

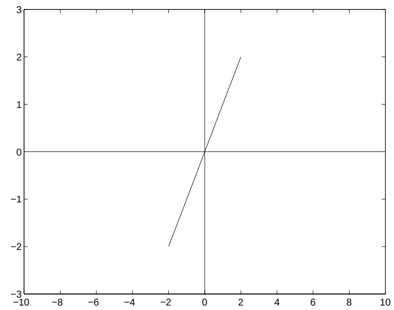


Figure 36: Graph of $f(x) = x$

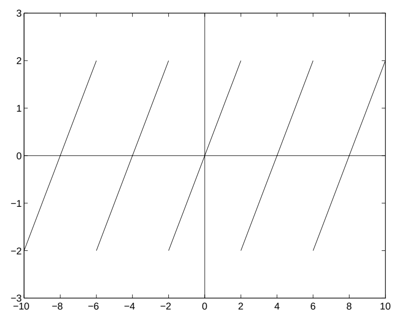


Figure 37: Graph of its periodic extension

2. a. $f(x) = x$

Since the periodic extension of $f(x)$ is discontinuous, the Fourier series is identical to (the periodic extension of) $f(x)$ everywhere except at the points of discontinuities. At those point $x = \pm L$ (and similar points in each period), we have

$$\frac{L + (-L)}{2} = 0$$

Now we evaluate the coefficients.

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx = \frac{1}{L} \int_{-L}^{L} x\, dx = 0$$

Since we have integrated an odd function on a symmetric interval. Similarly for all a_n.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x\, dx = \frac{1}{L} \left\{ \frac{-x \cos \frac{n\pi}{L} x}{\frac{n\pi}{L}} \Big|_{-L}^{L} + \int_{-L}^{L} \frac{\cos \frac{n\pi}{L} x}{\frac{n\pi}{L}}\, dx \right\}$$

This was a result of integration by parts.

$$= \frac{1}{L} \left\{ \frac{-L \cos n\pi}{\frac{n\pi}{L}} + \frac{-L \cos(-n\pi)}{\frac{n\pi}{L}} + \frac{\sin \frac{n\pi}{L} x}{\left(\frac{n\pi}{L}\right)^2} \Big|_{-L}^{L} \right\}$$

The last term vanishes at both end points $\pm L$

$$= \frac{1}{L}\frac{-2L\cos n\pi}{\frac{n\pi}{L}} = -\frac{2L}{n\pi}(-1)^n$$

Thus

$$b_n = \frac{2L}{n\pi}(-1)^{n+1}$$

and the Fourier series is

$$x \sim \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{L}x$$

2. b. This function is already in a Fourier sine series form and thus we can read the coefficients

$$a_n = 0 \qquad n = 0, 1, 2, \ldots$$
$$b_n = 0 \qquad n \neq 1$$
$$b_1 = 1$$

2. c.

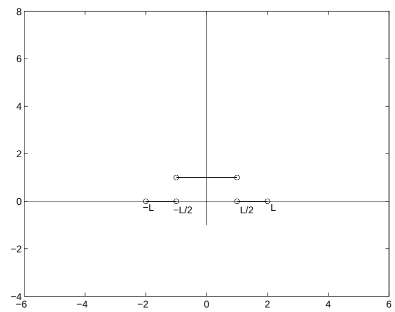


Figure 38: graph of $f(x)$ for problem 2c

Since the function is even, all the coefficients b_n will vanish.

$$a_0 = \frac{1}{L}\int_{-L/2}^{L/2} dx = \frac{1}{L} x|_{-L/2}^{L/2} = \frac{1}{L}\left(\frac{L}{2} - \left(-\frac{L}{2}\right)\right) = 1$$

$$a_n = \frac{1}{L}\int_{-L/2}^{L/2} \cos\frac{n\pi}{L}x\, dx = \frac{1}{L}\frac{L}{n\pi} \sin\frac{n\pi}{L}x|_{-L/2}^{L/2} = \frac{1}{n\pi}\left(\sin\frac{n\pi}{2} - \sin\frac{-n\pi}{2}\right)$$

Since the sine function is odd the last two terms add up and we have

$$a_n = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

The Fourier series is

$$f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\frac{n\pi}{2} \cos\frac{n\pi}{L}x$$

3.

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x \right)$$

$$g(x) \sim \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi}{L}x + B_n \sin \frac{n\pi}{L}x \right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi}{L}x \, dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi}{L}x \, dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} g(x) \cos \frac{n\pi}{L}x \, dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L} g(x) \sin \frac{n\pi}{L}x \, dx$$

For $\alpha f(x) + \beta g(x)$ we have

$$\frac{1}{2}\gamma_0 + \sum_{n=1}^{\infty} \left(\gamma_n \cos \frac{n\pi}{L}x + \delta_n \sin \frac{n\pi}{L}x \right)$$

and the coefficients are

$$\gamma_0 = \frac{1}{L}\int_{-L}^{L} (\alpha f(x) + \beta g(x)) dx$$

which by linearity of the integral is

$$\gamma_0 = \alpha \frac{1}{L}\int_{-L}^{L} f(x) dx + \beta \frac{1}{L}\int_{-L}^{L} g(x) dx = \alpha a_0 + \beta A_0$$

Similarly for γ_n and δ_n.

$$\gamma_n = \frac{1}{L}\int_{-L}^{L} (\alpha f(x) + \beta g(x)) \cos \frac{n\pi}{L}x \, dx$$

which by linearity of the integral is

$$\gamma_n = \alpha \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi}{L}x \, dx + \beta \frac{1}{L}\int_{-L}^{L} g(x) \cos \frac{n\pi}{L}x \, dx = \alpha a_n + \beta A_n$$

$$\delta_n = \frac{1}{L}\int_{-L}^{L} (\alpha f(x) + \beta g(x)) \sin \frac{n\pi}{L}x \, dx$$

which by linearity of the integral is

$$\delta_n = \alpha \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi}{L}x \, dx + \beta \frac{1}{L}\int_{-L}^{L} g(x) \sin \frac{n\pi}{L}x \, dx = \alpha b_n + \beta B_n$$

5.4 Relationship to Least Squares

5.5 Convergence

5.6 Fourier Cosine and Sine Series

Problems

1. For each of the following functions
   i. Sketch $f(x)$
   ii. Sketch the Fourier series of $f(x)$
   iii. Sketch the Fourier sine series of $f(x)$
   iv. Sketch the Fourier cosine series of $f(x)$
   a. $f(x) = \begin{cases} x & x < 0 \\ 1 + x & x > 0 \end{cases}$
   b. $f(x) = e^x$
   c. $f(x) = 1 + x^2$
   d. $f(x) = \begin{cases} \frac{1}{2}x + 1 & -2 < x < 0 \\ x & 0 < x < 2 \end{cases}$

2. Sketch the Fourier sine series of

$$f(x) = \cos \frac{\pi}{L} x.$$

Roughly sketch the sum of the first three terms of the Fourier sine series.

3. Sketch the Fourier cosine series and evaluate its coefficients for

$$f(x) = \begin{cases} 1 & x < \frac{L}{6} \\ \\ 3 & \frac{L}{6} < x < \frac{L}{2} \\ \\ 0 & \frac{L}{2} < x \end{cases}$$

4. Fourier series can be defined on other intervals besides $[-L, L]$. Suppose $g(y)$ is defined on $[a, b]$ and periodic with period $b - a$. Evaluate the coefficients of the Fourier series.

5. Expand

$$f(x) = \begin{cases} 1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

in a series of $\sin nx$.

   a. Evaluate the coefficients explicitly.

   b. Graph the function to which the series converges to over $-2\pi < x < 2\pi$.

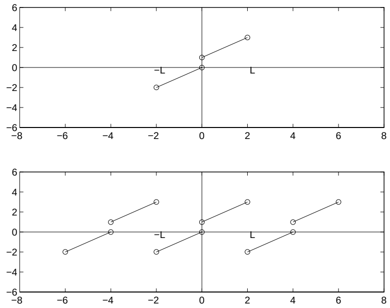


Figure 39: Sketch of $f(x)$ and its periodic extension for 1a

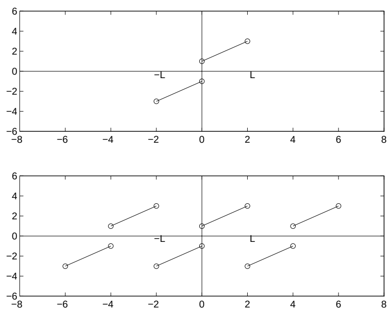


Figure 40: Sketch of the odd extension and its periodic extension for 1a

1. a.

The Fourier series is the same as the periodic extension except for the points of discontinuities where the Fourier series yields

$$\frac{1\,+\,0}{2}\;=\;\frac{1}{2}$$

For the Fourier sine series we take ONLY the right branch on the interval $[0, L]$ and extend it as an odd function.

Now the same discontinuities are there but the value of the Fourier series at those points is

$$\frac{1\,+\,(-1)}{2}\;=\;0$$

For the Fourier cosine series we need an even extension

Note that the periodic extension IS continuous and the Fourier series gives the exact same sketch.

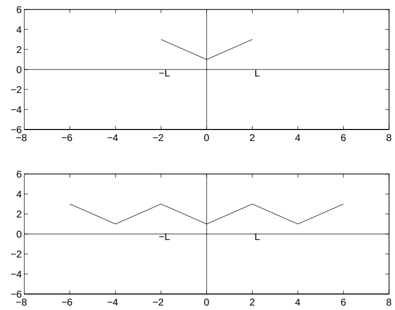


Figure 41: Sketch of the even extension and its periodic extension for 1a

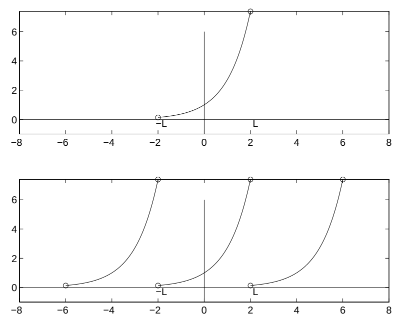


Figure 42: Sketch of $f(x)$ and its periodic extension for 1b

1. b.

The Fourier series is the same as the periodic extension except for the points of discontinuities where the Fourier series yields

$$\frac{e^L + e^{-L}}{2} = \cosh L$$

For the Fourier sine series we take ONLY the right branch on the interval $[0, L]$ and extend it as an odd function.

Now the same discontinuities are there but the value of the Fourier series at those points

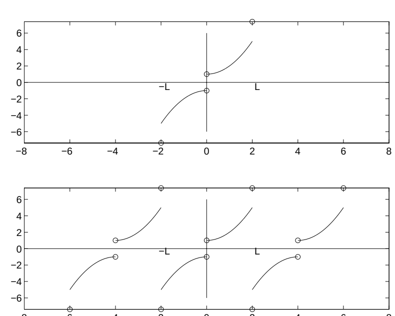


Figure 43: Sketch of the odd extension and its periodic extension for 1b

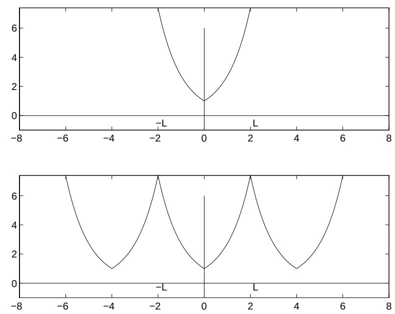


Figure 44: Sketch of the even extension and its periodic extension for 1b

is

$$\frac{1 + (-1)}{2} = 0$$

For the Fourier cosine series we need an even extension

Note that the periodic extension IS continuous and the Fourier series gives the exact same sketch.

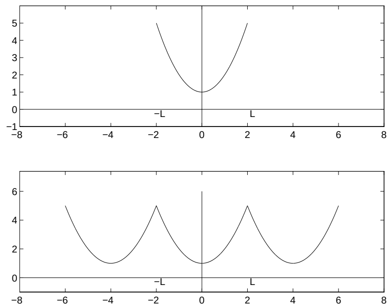


Figure 45: Sketch of $f(x)$ and its periodic extension for 1c

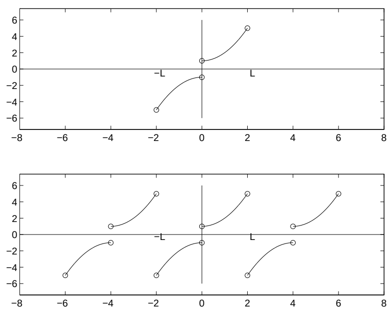


Figure 46: Sketch of the odd extension and its periodic extension for 1c

1. c.

The Fourier series is the same as the periodic extension. In fact the Fourier cosine series is the same!!!

For the Fourier sine series we take ONLY the right branch on the interval $[0, L]$ and extend it as an odd function.

Now the same discontinuities are there but the value of the Fourier series at those points is

$$\frac{1 + (-1)}{2} = 0$$

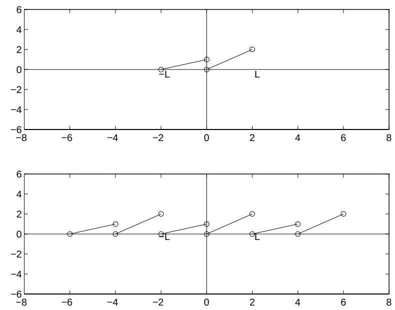


Figure 47: Sketch of $f(x)$ and its periodic extension for 1d

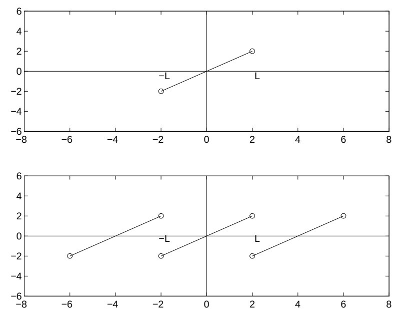


Figure 48: Sketch of the odd extension and its periodic extension for 1d

1. d.

The Fourier series is the same as the periodic extension except for the points of discontinuities where the Fourier series yields

$$\frac{1 + 0}{2} = \frac{1}{2} \qquad \text{for } x = 0 + \text{multiples of } 4$$

$$\frac{1 + 2}{2} = \frac{3}{2} \qquad \text{for } x = 2 + \text{multiples of } 4$$

For the Fourier sine series we take ONLY the right branch on the interval $[0, L]$ and extend it as an odd function.

Now some of the same discontinuities are there but the value of the Fourier series at those points is

$$\frac{2 + (-2)}{2} = 0 \qquad \text{for } x = 2 + \text{multiples of } 4$$

At the other previous discontinuities we now have continuity.

For the Fourier cosine series we need an even extension

Note that the periodic extension IS continuous and the Fourier series gives the exact same sketch.

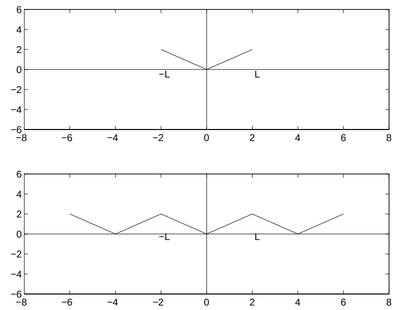


Figure 49: Sketch of the even extension and its periodic extension for 1d

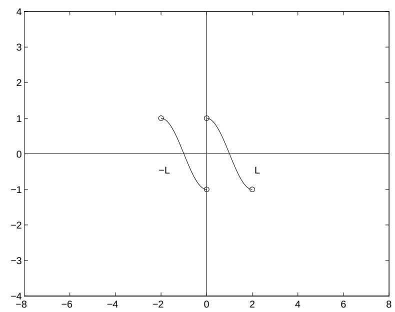


Figure 50: Sketch of the odd extension for 2

2.

$$\cos \frac{\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin \frac{n \pi}{L} x$$

$$bn = \begin{cases} 0 & n \,\text{odd} \\ \dfrac{4n}{\pi \left(n^2 - 1\right)} & n \,\text{even} \end{cases}$$

Since we have a Fourier sine series, we need the odd extension of $f(x)$

Now extend by periodicity
At points of discontinuity the Fourier series give zero.

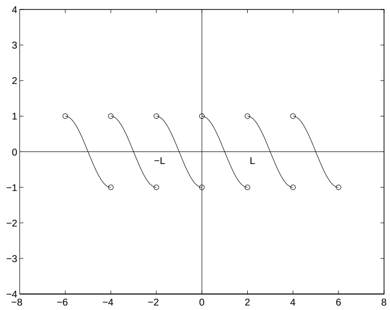


Figure 51: Sketch of the periodic extension of the odd extension for 2

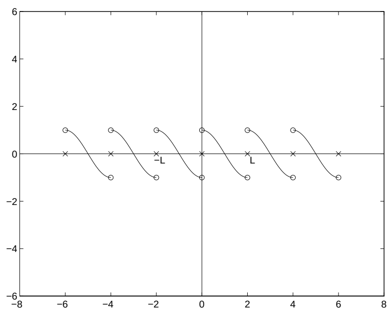


Figure 52: Sketch of the Fourier sine series for 2

First two terms of the Fourier sine series of $\cos \dfrac{\pi x}{L}$ are

$$= b_2 \, sin \, \frac{2\pi x}{L} + b_4 \sin \frac{4\pi x}{L}$$

$$= \frac{8}{3\pi} \sin \frac{2\pi}{L} x + \frac{16}{15\pi} \sin \frac{4\pi}{L} x$$

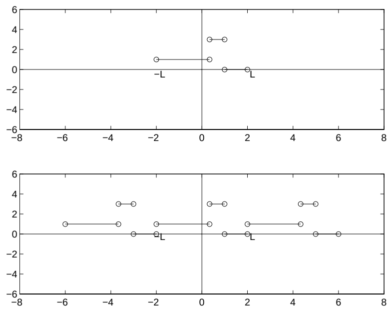


Figure 53: Sketch of $f(x)$ and its periodic extension for problem 3

3.

$$f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$$

Fourier cosine series coefficients:

$$a_0 = \frac{2}{L}\left\{\int_0^{L/6} dx + \int_{L/6}^{L/2} 3\,dx\right\} = \frac{2}{L}\left\{\frac{L}{6} + 3\left(\frac{L}{2} - \frac{L}{6}\right)\right\}$$

$$= \frac{2}{L}\left\{\frac{L}{6} + 3\,\frac{2L}{6}\right\} = \frac{2}{L}\cdot\frac{7}{6}L = \frac{7}{3}$$

$$\begin{aligned}
a_n &= \tfrac{2}{L}\left\{\int_0^{L/6} \cos \tfrac{n\pi}{L} x\, dx + 3\int_{L/6}^{L/2} \cos \tfrac{n\pi}{L} x\, dx\right\} \\
&= \tfrac{2}{L}\left\{\frac{\sin \frac{n\pi}{L} x}{\frac{n\pi}{L}}\Big|_0^{L/6} + 3\,\frac{\sin \frac{n\pi}{L} x}{\frac{n\pi}{L}}\Big|_{L/6}^{L/2}\right\} \\
&= \tfrac{2}{L}\left\{\frac{\sin \frac{n\pi}{6}}{\frac{n\pi}{L}} + 3\,\frac{\sin \frac{n\pi}{2} - \sin \frac{n\pi}{6}}{\frac{n\pi}{6}}\right\} \\
&= \tfrac{2}{L}\,\tfrac{L}{n\pi}\left\{3\underbrace{\sin \frac{n\pi}{2}}_{0 \text{ for n even}} - 2\sin \tfrac{n\pi}{6}\right\} = \tfrac{2}{n\pi}\left\{3\sin \tfrac{n\pi}{2} - 2\sin \tfrac{n\pi}{6}\right\}
\end{aligned}$$

± 1 for n odd

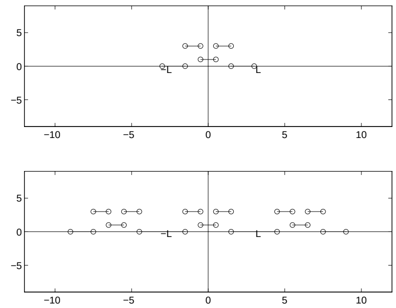


Figure 54: Sketch of the even extension of $f(x)$ and its periodic extension for problem 3

4.

$$x \,\epsilon\, [-L,\, L]$$

$$y \,\epsilon\, [a,\, b]$$

then $y = \frac{a+b}{2} + \frac{b-a}{2L} x$ $(*)$ is the transformation required (Note that if $x = -L$ then $y = a$ and if $x = L$ then $y = b$)

$g(y)$ is periodic of period $b - a$

$$g(y) = G(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x)$$

$$a_n = \frac{1}{L} \int_{-L}^{L} G(x) \cos \frac{n\pi}{L} x \, dx$$

solving (*) for x yields

$$x = \frac{2L}{b-a} \left[y - \frac{a+b}{2} \right] \quad \Rightarrow \quad dx = \frac{2L}{b-a} dy$$

$$a_n = \frac{1}{L} \int_a^b g(y) \cos \left(\frac{n\pi}{L} \frac{2L}{b-a} \left(y - \frac{a+b}{2} \right) \right) \frac{2L}{b-a} dy$$

Therefore

$$a_n = \frac{2}{b-a} \int_a^b g(y) \cos \left[\frac{2n\pi}{b-a} \left(y - \frac{a+b}{2} \right) \right] dy$$

Similarly for b_n

$$b_n = \frac{2}{b-a} \int_a^b g(y) \sin \left[\frac{2n\pi}{b-a} \left(y - \frac{a+b}{2} \right) \right] dy$$

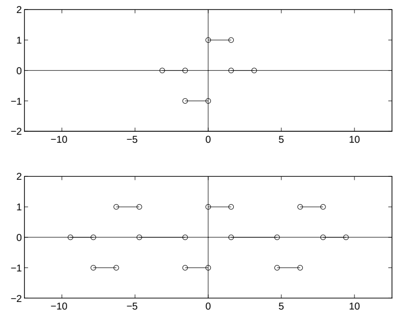


Figure 55: Sketch of the periodic extension of the odd extension of $f(x)$ (problem 5)

5.

$$f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 0 & \pi/2 < x < \pi \end{cases}$$

Expand in series of sin nx

$$f(x) \sim \sum_{n=1}^{\pi} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi/2} 1 \cdot \sin nx \, dx$$

$$\uparrow$$

$f(x) =$ zero on the rest

$$= \frac{2}{\pi} \left(-\frac{1}{n} \cos nx \right) \Big|_0^{\pi/2}$$

$$= -\frac{2}{n\pi} \underbrace{\cos \frac{n\pi}{2}}_{\uparrow} + \frac{2}{n\pi}$$

this takes the values 0, ± 1 depending on n!!!

5.7 Term by Term Differentiation

Problems

1. Given the Fourier sine series

$$\cos\frac{\pi}{L}x \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

$$b_n = \begin{cases} 0 & n \text{ is odd} \\ \frac{4n}{\pi(n^2-1)} & n \text{ is even} \end{cases}$$

Determine the Fourier cosine series of $\sin\frac{\pi}{L}x$.

2. Consider

$$\sinh x \sim \sum_{n=1}^{\infty} a_n \sin nx.$$

Determine the coefficients a_n by differentiating twice.

1. Fourier cosine series of $\sin \dfrac{\pi x}{L}$ using

$$\underbrace{\cos \frac{\pi x}{L}}_{f(x)} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

with

$$b_n = \begin{cases} 0 & \text{n odd} \\ \dfrac{4n}{\pi(n^2 - 1)} & \text{n even} \end{cases}$$

$$-\frac{\pi}{L} \sin \frac{\pi}{L} x \sim \frac{1}{L}[f(L) - f(0)] + \sum_{n=1}^{\infty} \left\{ \frac{n\pi}{L} b_n + \frac{2}{L}[(-1)^n f(L) - f(0)] \right\} \cos \frac{n\pi}{L} x$$

$$f(L) = \cos \pi = -1$$

$$f(0) = \cos 0 = 1$$

$$= -\frac{2}{L} + \sum_{n=1}^{\infty} \left\{ \frac{n\pi}{L} b_n + \frac{2}{L}[(-1)^{n+1} - 1] \right\} \cos \frac{n\pi}{L} x$$

$$\sin \frac{\pi x}{L} = -\frac{L}{\pi} \left\{ -\frac{2}{L} + \sum_{n=1}^{\infty} \left[\frac{n\pi}{L} b_n - \frac{2}{L} \left(\underbrace{(-1)^n + 1} \right) \right] \cos \frac{n\pi}{L} x \right.$$

$= 0$ if n is odd

$= 2$ if n is even

Substitute for b_n

$$\sin \frac{\pi}{L} x = \frac{2}{\pi} - \frac{4}{\pi} \underbrace{\sum_{n=2}^{\infty}}_{n \text{ even}} \left[\underbrace{\frac{n^2}{(n^2 - 1)} - 1}_{\frac{1}{n^2-1}} \right] \cos \frac{n\pi}{L} x$$

Check constant term $\frac{1}{2}\, a_0$ where

$$a_0 \;=\; \frac{2}{L}\int_0^L \sin\,\frac{\pi}{L}\,x\,dx \;=\; \frac{2}{L}\left(-\,\frac{L}{\pi}\,\cos\,\frac{\pi}{L}\,x\right)\,\Big|_0^L \;=\; -\,\frac{2}{\pi}\,(-1\;-1) \;=\; \frac{4}{\pi}\;\Rightarrow$$

$$\tfrac{1}{2}\,a_0 \;=\; \tfrac{2}{\pi}$$

This agrees with previous result.

2. $\sinh x \sim \sum_{n=1}^{\infty} a_n \sin nx$

Differentiate (since this is Fourier sine series and $\sinh \pi \neq 0$, we have to use the formula)

$$\cosh x \sim \frac{1}{\pi}(\sinh \pi - \sinh 0) + \sum_{n=1}^{\infty} \left\{ na_n + \frac{2}{\pi}\left[(-1)^n \sinh \pi - \sinh 0\right] \right\} \cos nx$$

Differentiate again (this time we have Fourier cosine series)

$$\sinh x \sim 0 + \sum_{n=1}^{\infty} (-n) \left\{ na_n + \frac{2}{\pi} (-1)^n \sinh \pi \right] \sin n\, x$$

$$a_n = -n^2 a_n + (-1)^{n+1} n \frac{2}{\pi} \sinh \pi$$

$$\boxed{a_n = \frac{(-1)^{n+1} n \frac{2}{\pi} \sinh \pi}{1 + n^2}}$$

If we go thru integration

$a_n = \frac{2}{\pi} \int_0^{\pi} \sinh x \sin nx \, dx$

we get the same answer.

5.8 Term by Term Integration

Problems

1. Consider

$$x^2 \sim \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x$$

a. Determine a_n by integration of the Fourier sine series of $f(x) = 1$, i.e. the series

$$1 \sim \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{2n-1}{L} \pi x$$

b. Derive the Fourier cosine series of x^3 from this.

2. Suppose that

$$\cosh x \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

a. Determine the coefficients b_n by differentiating twice.
b. Determine b_n by integrating twice.

3. Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

by using the integration of Fourier sine series of $f(x) = 1$ (see problem 1 part a.)

1a. $\quad 1 = \dfrac{4}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{2n-1} \sin \dfrac{2n-1}{L} \pi x$

Integrate

$$x = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \frac{-L}{(2n-1)\pi} \cos \frac{2n-1}{L} \pi x \,\Big|_0^x$$

$$= -\frac{4L}{\pi^2} \sum_{n=1} \frac{1}{(2n-1)^2} \cos \frac{2n-1}{L} \pi x + \underbrace{\frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}}_{\text{constant term}}$$

To find the constant term $\frac{1}{2} a_0$

$$a_0 = \frac{2}{L} \int_0^L x \, dx = \frac{2}{2L} x^2 \,\Big|_0^L = \frac{L^2}{L} = L$$

$$\boxed{x = \frac{L}{2} - \frac{4L}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{2n-1}{L} \pi x}$$

Integrate again

$$\frac{x^2}{2} = \frac{L}{2} x - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \frac{L}{(2n-1)^3} \sin \frac{2n-1}{L} \pi x \,\Big|_0^x$$

$$\boxed{\frac{x^2}{2} = \frac{L}{2} x - \frac{4L^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{2n-1}{L} \pi x}$$

We need Fourier sine series of x to complete the work

See 2a in Section 5.3

$$\boxed{x = \sum_{n=1}^{\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin \frac{n\pi}{L} x}$$

$$x^2 = L \sum_{n=1}^{\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin \frac{n\pi}{L} x - \frac{8L^2}{\pi^3} \underbrace{\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{2n-1}{L} \pi x}_{\sum_{n=1,3,\cdots} \frac{1}{n^3} \sin \frac{n}{L} \pi x}$$

$$x^2 = \sum a_n \sin \frac{n\,\pi}{L}\, x$$

where (only for odd n we get contribution from <u>both</u> sums)

$$a_1 = \frac{2L^2}{\pi} - \frac{8L^2}{\pi^3}$$

$$a_2 = \frac{2L^2}{2\pi}$$

$$a_3 = \frac{2L^2}{3\pi} - \frac{8L^2}{\pi\,3} \cdot \frac{1}{3^3}$$

and so on.

1b. $x^2 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x$

To get the cosine series for x^3, let's integrate

$\frac{x^3}{3} + C = -\sum_{n=1}^{\infty} a_n \frac{\cos \frac{n\pi}{L} x}{\frac{n\pi}{L}}$

Integrate again on [0, L] the integral of $\cos \frac{n\pi}{L} x$ will give zero $\Rightarrow$

$$\int_0^L \frac{x^3}{3}\, dx + C\, x|_0^L = 0$$

$$\frac{x^4}{12}|_0^L + C\, L = 0$$

$$\frac{L^4}{12} + C\, L = 0$$

$$C = -\frac{L^3}{12}$$

$$\Rightarrow \qquad x^3 = \frac{L^3}{4} - \frac{3L}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{n} \cos \frac{n\pi}{L} x$$

where a_n as in 1a

This problem can be done as in 1a.
Integrate the series from 0 to x

$\frac{x^3}{3} = \sum_{n=1}^{\infty} a_n \frac{1}{\frac{n\pi}{L}} - \sum_{n=1}^{\infty} a_n \frac{\cos \frac{n\pi}{L} x}{\frac{n\pi}{L}}$

Multiply by 3

$x^3 = \underbrace{\sum_{n=1}^{\infty} a_n \frac{3}{\frac{n\pi}{L}}}_{\text{constant term}=1/2a_0} - 3\sum_{n=1}^{\infty} a_n \frac{\cos \frac{n\pi}{L} x}{\frac{n\pi}{L}}$

To find the constant term we evaluate diredtly

$$a_0 = \frac{2}{L}\int_0^L x^3 dx = \frac{2}{L}\frac{x^4}{4}|_0^L = \frac{2}{L}\frac{L^4}{4} = \frac{L^3}{2}$$

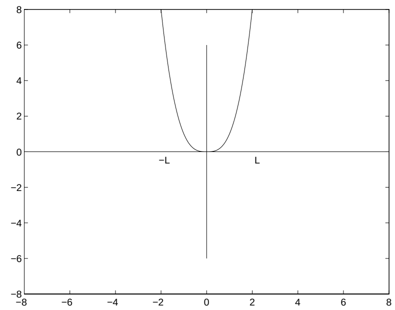


Figure 56: Sketch of the even extension of $f(x) = x^3$ (problem 1b)

2. $$\underbrace{\cosh\, x}_{\text{even function on [-L, L]}} \sim \sum_{n=1}^{\infty} b_n \, \sin \, \frac{n\,\pi}{L}\, x$$

To get sine series we must take the odd extension

a. $$\sinh\, x \sim \underbrace{\frac{1}{L}\left[\cosh\, L - \underbrace{\cosh\, 0}_{=1}\right]}_{\text{constant}} + \sum_{n=1}^{\infty}\left(\frac{n\,\pi}{L}\, b_n + \frac{2}{L}\left[(-1)^n \cosh\, L - \underbrace{\cosh\, 0}_{=1}\right]\right) \cos\, \frac{n\,\pi}{L}\, x$$

Differentiate again

$$\cosh\, x \sim \quad 0 + \sum_{n=1}^{\infty}\left(\frac{n\,\pi}{L}\, b_n + \frac{2}{L}\, [(-1)^n \cosh\, L - 1]\right) (-\frac{n\,\pi}{L} \, \sin\, \frac{n\,\pi}{L}\, x)$$

$$= \sum_{n=1}^{\infty}\left(-\left(\frac{n\,\pi}{L}\right)^2 b_n - \frac{2n\,\pi}{L^2}\, [(-1)^n \cosh\, L - 1]\right) \sin\, \frac{n\,\pi}{L} x$$

$$\Rightarrow \qquad b_n = -\left(\frac{n\,\pi}{L}\right)^2 b_n - \frac{2n\,\pi}{L^2}\, [(-1)^n \cosh\, L - 1]$$

$\Rightarrow$

$$\boxed{b_n = \frac{-\frac{2n\,\pi}{L^2}\, [(-1)^n \cosh\, L - 1]}{1 + \left(\frac{n\,\pi}{L}\right)^2}}$$

b. Integrations

first

$$\sinh x = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(-\frac{L}{n\pi}\right) b_n \cos \frac{n\pi}{L} x$$

where $\quad A_0 = \frac{2}{L} \int_0^L \sinh x \, dx = \frac{2}{L} (\cosh L - 1)$

second

$$\cosh x - 1 = A_0 x + \sum_{n=1}^{\infty} \left[-\left(\frac{L}{n\pi}\right)^2 b_n \sin \frac{n\pi}{L} x \right]$$

$\Rightarrow \qquad \sum b_n \, [1 + \left(\frac{L}{n\pi}\right)^2] \sin \frac{n\pi}{L} x = 1 + A_0 x$

Note (Done previously)

$$x \sim \sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{L} x$$

where

$$c_n = \frac{2L}{n\pi} (-1)^{n+1}$$

and

$$1 \sim \sum d_n \sin \frac{n\pi}{L} x$$

where

$$d_n = \begin{cases} 0 & \text{n even} \\ \dfrac{4}{n\pi} & \text{n odd} \end{cases} = \frac{2}{n\pi} [1 - (-1)^n]$$

$\Rightarrow \qquad b_n [1 + \left(\frac{L}{n\pi}\right)^2] = \frac{2}{n\pi} [1 - (-1)^n] + \frac{1}{L} [\cosh L - 1] \frac{2L}{n\pi} (-1)^{n+1}$

which yields the same answer after longer computations.

3. Evaluate

$$1 + \frac{1}{3^2} + \cdots$$

from 1a:

$$x = \frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\frac{\pi\, x}{L} + \frac{\cos\frac{3\pi}{L}x}{3^2} + \cdots\right)$$

<u>at $x = 0$</u>

$$0 = \frac{L}{2} - \frac{4L}{\pi^2}\left(1 + \frac{1}{3^2} + \cdots\right)$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{L}{2} \cdot \frac{\pi^2}{4L} = \underline{\frac{\pi^2}{8}}$$

5.9 Full solution of Several Problems

Problems

1. Solve the heat equation

$$u_t = k u_{xx}, \qquad 0 < x < L, \qquad t > 0,$$

subject to the boundary conditions

$$u(0,t) = u(L,t) = 0.$$

Solve the problem subject to the initial value:

a. $u(x,0) = 6 \sin \frac{9\pi}{L} x$.

b. $u(x,0) = 2 \cos \frac{3\pi}{L} x$.

2. Solve the heat equation

$$u_t = k u_{xx}, \qquad 0 < x < L, \qquad t > 0,$$

subject to

$$u_x(0,t) = 0, \qquad t > 0$$
$$u_x(L,t) = 0, \qquad t > 0$$

a. $u(x,0) = \begin{cases} 0 & x < \frac{L}{2} \\ 1 & x > \frac{L}{2} \end{cases}$

b. $u(x,0) = 6 + 4 \cos \frac{3\pi}{L} x$.

3. Solve the eigenvalue problem

$$\phi'' = -\lambda \phi$$

subject to

$$\phi(0) = \phi(2\pi)$$
$$\phi'(0) = \phi'(2\pi)$$

4. Solve Laplace's equation inside a wedge of radius a and angle α,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

subject to

$$u(a,\theta) = f(\theta),$$
$$u(r,0) = u_\theta(r,\alpha) = 0.$$

5. Solve Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$ subject to

a. $u_x(0,y) = u_x(L,y) = u(x,0) = 0, \quad u(x,H) = f(x).$

b. $u(0,y) = g(y), \quad u(L,y) = u_y(x,0) = u(x,H) = 0.$

c. $u(0,y) = u(L,y) = 0, \quad u(x,0) - u_y(x,0) = 0, \quad u(x,H) = f(x).$

6. Solve Laplace's equation outside a circular disk of radius a, subject to

a. $u(a,\theta) = \ln 2 + 4\cos 3\theta.$

b. $u(a,\theta) = f(\theta).$

7. Solve Laplace's equation inside the quarter circle of radius 1, subject to

a. $u_\theta(r,0) = u(r,\pi/2) = 0, \qquad u(1,\theta) = f(\theta).$

b. $u_\theta(r,0) = u_\theta(r,\pi/2) = 0, \qquad u_r(1,\theta) = g(\theta).$

c. $u(r,0) = u(r,\pi/2) = 0, \qquad u_r(1,\theta) = 1.$

8. Solve Laplace's equation inside a circular annulus ($a < r < b$), subject to

a. $u(a,\theta) = f(\theta), \qquad u(b,\theta) = g(\theta).$

b. $u_r(a,\theta) = f(\theta), \qquad u_r(b,\theta) = g(\theta).$

9. Solve Laplace's equation inside a semi-infinite strip ($0 < x < \infty, \quad 0 < y < H$) subject to

$u_y(x,0) = 0, \qquad u_y(x,H) = 0, \qquad u(0,y) = f(y).$

10. Consider the heat equation

$$u_t = u_{xx} + q(x,t), \qquad\qquad 0 < x < L,$$

subject to the boundary conditions

$$u(0,t) = u(L,t) = 0.$$

Assume that $q(x,t)$ is a piecewise smooth function of x for each positive t. Also assume that u and u_x are continuous functions of x and u_{xx} and u_t are piecewise smooth. Thus

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi}{L} x.$$

Write the ordinary differential equation satisfied by $b_n(t)$.

11. Solve the following inhomogeneous problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + e^{-t} + e^{-2t}\cos\frac{3\pi}{L}x,$$

subject to

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0,$$
$$u(x,0) = f(x).$$

Hint : Look for a solution as a Fourier cosine series. Assume $k \neq \frac{2L^2}{9\pi^2}$.

12. Solve the wave equation by the method of separation of variables

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < L,$$

$$u(0,t) = 0,$$
$$u(L,t) = 0,$$
$$u(x,0) = f(x),$$
$$u_t(x,0) = g(x).$$

13. Solve the heat equation

$$u_t = 2u_{xx}, \qquad 0 < x < L,$$

subject to the boundary conditions

$$u(0,t) = u_x(L,t) = 0,$$

and the initial condition

$$u(x,0) = \sin \frac{3}{2}\frac{\pi}{L}x.$$

14. Solve the heat equation

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

inside a disk of radius a subject to the boundary condition

$$\frac{\partial u}{\partial r}(a,\theta,t) = 0,$$

and the initial condition

$$u(r,\theta,0) = f(r,\theta)$$

where $f(r,\theta)$ is a given function.

1 a. $u_t = ku_{xx}$

$u(0, t) = 0$

$u(L, t) = 0$

$u(x, 0) = 6 \sin \dfrac{9\pi x}{L}$

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t}$$

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \equiv 6 \sin \frac{9\pi x}{L}$$

$\Rightarrow$ the only term from the sum that can survive is for $n = 9$ with $B_9 = 6$ $B_n = 0$ for $n \neq 9$

$$\boxed{\Rightarrow u(x, t) = 6 \sin \frac{9\pi x}{L} e^{-k(\frac{9\pi}{L})^2 t}}$$

b. $u(x, 0) = 2 \cos \dfrac{3\pi x}{L}$

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t}$$

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = 2 \cos \frac{3\pi x}{L}$$

$$\Rightarrow \quad B_n = \frac{2}{L} \int_0^L 2 \cos \frac{3\pi x}{L} \sin \frac{n\pi x}{L} dx$$

compute the integral for $n = 1, 2, \ldots$ to get B_n.

To compute the coefficients, we need the integral

$$\int_0^L \cos\frac{3\pi}{L}x\,\sin\frac{n\pi}{L}x\,dx$$

Using the trigonometric identity

$$\sin a\,\cos b = \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right)$$

we have

$$\frac{1}{2}\int_0^L \left(\sin\frac{(n+3)\pi}{L}x + \sin\frac{(n-3)\pi}{L}x\right)dx$$

Now for $n \neq 3$ the integral is

$$-\frac{1}{2}\frac{\cos\frac{(n+3)\pi}{L}x}{\frac{(n+3)\pi}{L}}\Big|_0^L - \frac{1}{2}\frac{\cos\frac{(n-3)\pi}{L}x}{\frac{(n-3)\pi}{L}}\Big|_0^L$$

or when recalling that $\cos m\pi = (-1)^m$

$$-\frac{L}{2\pi(n+3)}\left[(-1)^{n+3} - 1\right] - \frac{L}{2\pi(n-3)}\left[(-1)^{n-3} - 1\right], \qquad \text{for } n \neq 3$$

Note that for n odd, the coefficient is zero.

For $n = 3$ the integral is

$$\int_0^L \cos\frac{3\pi}{L}x\,\sin\frac{3\pi}{L}x\,dx = \frac{1}{2}\int_0^L \sin\frac{6\pi}{L}x\,dx$$

which is

$$-\frac{1}{2}\frac{L}{6\pi}\cos\frac{6\pi}{L}x\,\Big|_0^L = 0$$

2. $u_t = k u_{xx}$

$u_x(0,\, t) = 0$

$u_x(L,\, t) = 0$

$u(x,\, 0) = f(x)$

$u(x,\, t) = X(x)\, T(t)$

$\dot{T} x = k X''\, T$

$\dfrac{\dot{T}}{kT} = \dfrac{X''}{x} = -\lambda$

$\dot{T} + \lambda\, k\, T = 0$

$$\left\{ \begin{array}{l} X'' + \lambda\, X = 0 \\ \\ X'(0) = 0 \\ \\ X'(L) = 0 \end{array} \right\} \Rightarrow \begin{array}{l} X_n = A_n \cos \dfrac{n\,\pi\, x}{L}, \qquad n = 1, 2, \ldots \\ \lambda_n = \left(\dfrac{n\,\pi}{L}\right)^2, \qquad n = 1, 2, \ldots \\ \lambda_0 = 0 \qquad X_0 = A_0 \end{array}$$

$\dot{T}_n + \left(\dfrac{n\,\pi}{L}\right)^2 k T_n = 0$

$T_n = B_n\, e^{-(\frac{n\,\pi}{L})^2\, kt}$

$$u(x,\, t) = \underbrace{A_0\, B_0}_{=a_0} + \sum_{n=1}^{\infty} A_n \cos \frac{n\,\pi\, x}{L}\, B_n\, e^{-(\frac{n\,\pi}{L})^2\, kt}$$

$$u(x,\, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\,\pi\, x}{L}\, e^{-(\frac{n\,\pi}{L})^2\, kt}$$

a.

$$f(x) = \begin{cases} 0 & x < L/2 \\ \\ 1 & x > L/2 \end{cases}$$

$$u(x,\,0) = f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\,\pi}{L}\,x$$

$$a_0 = \frac{2}{2L}\int_{L/2}^{L} dx = \frac{1}{L}\left(L - \frac{L}{2}\right) = \frac{1}{2}$$

$$a_n = \frac{2}{L}\int_{L/2}^{L} \cos \frac{n\,\pi}{L}\,x\,dx = \frac{2}{L}\,\frac{L}{n\,\pi}\,\sin \frac{n\,\pi}{L}\,x\,\Big|_{L/2}^{L} = -\frac{2}{n\,\pi}\sin\frac{n\,\pi}{2}$$

$$\boxed{u(x,\,t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{2}{n\,\pi}\,\sin\,\frac{n\,\pi}{2}\right)\cos\frac{n\,\pi}{L}\,x\,e^{-k(\frac{n\,\pi}{L})^2\,t}}$$

b.

$$f(x) = 6 + 4\,\cos\frac{3\pi}{L}\,x$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L}\,x$$

$$a_0 = 6$$

$$a_3 = 4 \qquad a_n = 0 \qquad n \neq 3$$

$$\boxed{u(x,\,t) = 6 + 4\,\cos\tfrac{3\pi}{L}\,x\,e^{-k(\frac{3\,\pi}{L})^2\,t}}$$

3.

$$\phi'' + \lambda \phi = 0$$

$$\phi(0) = \phi(2\pi)$$

$$\phi'(0) = \phi'(2\pi)$$

$\underline{\lambda > 0}$ $\quad \phi = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x$

$$\phi' = -A\sqrt{\lambda} \sin \sqrt{\lambda} x + B\sqrt{\lambda} \cos \sqrt{\lambda} x$$

$$\phi(0) = \phi(2\pi) \Rightarrow A = A \cos 2\pi\sqrt{\lambda} + B \sin 2\pi\sqrt{\lambda}$$

$$\phi'(0) = \phi'(2\pi) \Rightarrow B\sqrt{\lambda} = -A\sqrt{\lambda} \sin 2\pi\sqrt{\lambda} + B\sqrt{\lambda} \cos 2\pi\sqrt{\lambda}$$

$$A(1 - \cos 2\pi\sqrt{\lambda}) - B \sin 2\pi\sqrt{\lambda} = 0$$

$$A\sqrt{\lambda} \sin 2\pi\sqrt{\lambda} + B\sqrt{\lambda}(1 - \cos 2\pi\sqrt{\lambda}) = 0$$

A system of 2 homogeneous equations. To get a nontrivial solution one must have the determinant $= 0$.

$$\begin{vmatrix} 1 - \cos 2\pi\sqrt{\lambda} & -\sin 2\pi\sqrt{\lambda} \\ \sqrt{\lambda} \sin 2\pi\sqrt{\lambda} & \sqrt{\lambda}(1 - \cos 2\pi\sqrt{\lambda}) \end{vmatrix} = 0$$

$$\sqrt{\lambda}\,(1 - \cos 2\pi\sqrt{\lambda})^2 + \sqrt{\lambda} \sin^2 2\pi\sqrt{\lambda} = 0$$

$$\sqrt{\lambda}\,\{1 - 2\cos 2\pi\sqrt{\lambda} + \underbrace{\cos^2 2\pi\sqrt{\lambda} + \sin^2 2\pi\sqrt{\lambda}}_{1}\} = 0$$

$$2\sqrt{\lambda}\{1 - \cos 2\pi\sqrt{\lambda}\} = 0 \quad \Rightarrow \quad \sqrt{\lambda} = 0 \quad \text{or} \quad \cos 2\pi\sqrt{\lambda} = 1$$

$$2\pi\sqrt{\lambda} = 2\pi n \quad n = 1, 2, \ldots$$

Since λ should be positive $\boxed{\lambda = n^2}$ $n = 1, 2, \ldots$

$$\lambda_n = n^2 \qquad \phi_n = A_n \cos nx + B_n \sin nx$$

$\underline{\lambda = 0}$ $\phi = Ax + B$

$\phi' = A$

$$\phi(0) = \phi(2\pi) \quad \Rightarrow \quad B = 2\pi A + B \quad \Rightarrow \quad A = 0$$

$$\phi'(0) = \phi'(2\pi) \quad \Rightarrow \quad A = A$$

$\Rightarrow$ $\underline{\lambda = 0}$ $\underline{\phi = B}$

$\underline{\lambda < 0}$ $\phi = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$

$$\phi(0) = \phi(2\pi) \quad \Rightarrow \quad A + B = Ae^{\sqrt{-\lambda}2\pi} + Be^{-2\pi\sqrt{-\lambda}}$$

$$\phi'(0) = \phi'(2\pi) \quad \Rightarrow \quad \sqrt{-\lambda}A - \sqrt{-\lambda}B = \sqrt{-\lambda}Ae^{\sqrt{-\lambda}2\pi} - \sqrt{-\lambda}\,Be^{-2\pi\sqrt{-\lambda}}$$

$$A\,[1 - e^{2\pi\sqrt{-\lambda}}] + B\,[1 - e^{-2\pi\sqrt{-\lambda}}] = 0$$

$$\sqrt{-\lambda}\,A\left[1 - e^{2\pi\sqrt{-\lambda}}\right] - B\sqrt{-\lambda}\left[1 - e^{-2\pi\sqrt{-\lambda}}\right] = 0$$

$$\begin{vmatrix} 1 - e^{2\pi\sqrt{-\lambda}} & 1 - e^{-2\pi\sqrt{-\lambda}} \\ \sqrt{-\lambda}\,(1 - e^{2\pi\sqrt{-\lambda}}) & -\sqrt{-\lambda}\,(1 - e^{-2\pi\sqrt{-\lambda}}) \end{vmatrix} = 0$$

$$-\sqrt{-\lambda}\,(1 - e^{2\pi\sqrt{-\lambda}})\,(1 - e^{-2\pi\sqrt{-\lambda}}) - \sqrt{-\lambda}(1 - e^{2\pi\sqrt{-\lambda}})\,(1 - e^{-2\pi\sqrt{-\lambda}}) = 0$$

$$-2\sqrt{-\lambda}(1 - e^{2\pi\sqrt{-\lambda}})\,(1 - e^{-2\pi\sqrt{-\lambda}}) = 0$$

$$1 - e^{2\pi\sqrt{-\lambda}} = 0 \qquad \text{or} \qquad 1 - e^{-2\pi\sqrt{-\lambda}} = 0$$

$$e^{2\pi\sqrt{-\lambda}} = 1 \qquad\qquad e^{-2\pi\sqrt{-\lambda}} = 1$$

Take ln of both sides

$$2\pi\sqrt{-\lambda} = 0 \qquad\qquad -2\pi\sqrt{-\lambda} = 0$$

$$\sqrt{-\lambda} = 0 \qquad\qquad \sqrt{-\lambda} = 0$$

not possible $\qquad\qquad$ not possible

Thus trivial solution if $\lambda < 0$

4. $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$u(a, \theta) = f(\theta)$$

$$u(r, 0) = u_\theta(r, \alpha) = 0$$

$$u(r, \theta) = R(r)\,\Theta(\theta)$$

$$\Theta\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{r^2}R\frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

multiply by $\dfrac{r^2}{R\Theta}$

$$\frac{r}{R}(r\,R')' = -\frac{\Theta''}{\Theta} = \lambda$$

$$\begin{cases} \Theta'' + \lambda\,\Theta = 0 & \rightarrow \quad \Theta(0) = \Theta'(\alpha) = 0 \\ r\,(r\,R')' - \lambda\,R = 0 & \rightarrow \quad |\,R(0)\,| < \infty \end{cases}$$

$$\Theta'' + \mu\,\Theta = 0 \qquad\qquad r(r\,R')' - \mu\,R = 0$$

$$\Theta\,(0) = 0 \qquad\qquad |\,R(0)\,| < \infty$$

$$\Theta'\,(\alpha) = 0 \qquad\qquad R_n = r^{(n-\frac{1}{2})\frac{\pi}{\alpha}}$$

$$\Downarrow \qquad\qquad \text{only positive exponent}$$

$$\Theta_n = \sin\,(n - 1/2)\,\frac{\pi}{\alpha}\,\theta \qquad\qquad \text{because of boundedness}$$

$$\mu_n = \left[(n - 1/2)\,\frac{\pi}{\alpha}\right]^2$$

$$n = 1,\, 2,\, \ldots$$

$$\boxed{u\,(r,\,\theta) = \sum_{n=1}^{\infty} a_n\, r^{(n-1/2)\pi/\alpha}\, \sin\,\frac{n-1/2}{\alpha}\,\pi\,\theta}$$

$$f(\theta) = \sum_{n=1}^{\infty} a_n\, a^{(n-1/2)\pi/\alpha}\, \sin\,\frac{(n - 1/2)\pi}{\alpha}\,\theta$$

$$\boxed{a_n = \frac{\int_0^{\alpha} f(\theta)\, \sin\,(n - 1/2)\,\frac{\pi}{\alpha}\,\theta\,d\,\theta}{a^{(n-1/2)\pi/\alpha}\,\int_0^{\alpha} \sin^2\,(n - 1/2)\,\frac{\pi}{\alpha}\,\theta\,d\,\theta}}$$

5. $u_{xx} + u_{yy} = 0$

$u_x(0, y) = 0$

$u_x(L, y) = 0$

$u(x, 0) = 0$

$u(x, H) = f(x)$

$u(x, y) = X(x)Y(y) \quad \Longrightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$

$u_x(0, y) = 0 \quad \Rightarrow \quad X'(0) = 0$

$u_x(L, y) = 0 \quad \Rightarrow \quad X'(L) = 0$

$u(x, 0) = 0 \quad \Rightarrow \quad Y(0) = 0$

$$\Rightarrow \quad \begin{array}{ll} X'' + \lambda X = 0 & Y'' - \lambda Y = 0 \\ X'(0) = 0 & Y(0) = 0 \\ X'(L) = 0 & \end{array}$$

$\Downarrow$ Table at the end of Chapter 4

$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 0, 1, 2, \ldots$

$x_n = \cos\frac{n\pi}{L}x \quad \Rightarrow \quad Y_n'' - \left(\frac{n\pi}{L}\right)^2 Y_n = 0 \qquad n = 0, 1, 2, \ldots$

If $n = 0 \Rightarrow Y_0'' = 0 \Rightarrow Y_0 = A_0 y + B_0$

$Y_0(0) = 0 \Rightarrow B_0 = 0$

$\Rightarrow \underline{Y_0(y) = A_0 y}$

If $n \neq 0 \Rightarrow Y_n = A_n e^{\left(\frac{n\pi}{L}\right)^2 y} + B_n e^{-\left(\frac{n\pi}{L}\right)^2 y}$

or

$Y_n = C_n \sinh\left(\frac{n\pi}{L} y + D_n\right)$

$Y_n(0) = 0 \Rightarrow D_n = 0$

$\Rightarrow \underline{Y_n = C_n \sinh\frac{n\pi}{L} y}$

$$u(x,\, y) \;=\; \underbrace{\alpha_0\, A_0}_{\frac{a_0}{2}}\, y \,\cdot\, 1 \,+\, \sum_{n=1}^{\infty} \underbrace{\alpha_n\, C_n}_{a_n} \cos\, \frac{n\,\pi}{L}\, x \,\sinh\, \frac{n\,\pi}{L}\, y$$

$$u(x,\, H) \;=\; \frac{a_0}{2} \,+\, \sum_{n=1}^{\infty} a_n \,\sinh\, \frac{n\,\pi}{L}\, H \,\cos\, \frac{n\,\pi}{L}\, x \;\equiv\; f(x)$$

This is the Fourier cosine series of $f(x)$

$$a_0\, H \;=\; \frac{2}{L} \int_0^L f(x)\, dx$$

$$a_n \,\sinh\, \frac{n\,\pi}{L}\, H \;=\; \frac{2}{L} \int_0^L f(x)\, \cos\, \frac{n\,\pi}{L}\, x\, dx$$

$$\Rightarrow \quad a_0 \;=\; \frac{2}{H L} \int_0^L f(x)\, dx$$

$$a_n \;=\; \frac{2}{L\, \sinh\, \frac{n\,\pi}{L}\, H} \int_0^L f(x)\, \cos\, \frac{n\,\pi}{L}\, x\, dx$$

and:

$$u(x,\, y) \;=\; \frac{a_0}{2}\, y \,+\, \sum_{n=1}^{\infty} a_n \,\sinh\, \frac{n\,\pi}{L}\, y \,\cos\, \frac{n\,\pi}{L}\, x$$

5 b. $u_{xx} + u_{yy} = 0$

$u(0, y) = g(y)$

$u(L, y) = 0$

$u_y(x, 0) = 0$

$u(x, H) = 0$

$X'' - \lambda X = 0 \qquad Y'' + \lambda Y = 0$

$X(L) = 0 \qquad Y'(0) = 0$

$\qquad\qquad Y(H) = 0$

Using the summary of Chapter 4 we have

$$Y_n(y) = \cos \frac{(n + \frac{1}{2})\pi}{H} y, \qquad n = 0, 1, \dots$$

$$\lambda_n = \left[\frac{\left(n + \frac{1}{2}\right) \pi}{H} \right]^2 \qquad n = 0, 1, \dots$$

Now use these eigenvalues in the x equation: $X_n'' - \left[\frac{(n+\frac{1}{2})\pi}{H}\right]^2 X_n = 0 \qquad n = 0, 1, 2, \dots$

Solve:

$X_n = c_n \sinh\left(\left(n + \frac{1}{2}\right) \frac{\pi}{H} x + D_n\right)$

Use the boundary condition: $X_n(L) = 0$

$X_n(L) = c_n \sinh\left(\left(n + \frac{1}{2}\right) \frac{\pi L}{H} + D_n\right) = 0 \Rightarrow \frac{\left(n+\frac{1}{2}\right)\pi L}{H} + D_n = 0$

$\Rightarrow X_n = c_n \sinh\left(\frac{(n+\frac{1}{2})\pi}{H}(x - L)\right)$

$$\boxed{\Rightarrow u(x, y) = \sum_{n=0}^{\infty} a_n \sinh\left[\frac{(n + \frac{1}{2})\pi}{H}(x - L)\right] \cos\left(n + \frac{1}{2}\right) \frac{\pi}{H} y}$$

To find the coefficients a_n, we use the inhomogeneous boundary condition:

$$u(0, y) = g(y) = \sum_{n=0}^{\infty} a_n \sinh\left(\frac{(n + \frac{1}{2})\pi}{H}(-L)\right) \cos\left(n + \frac{1}{2}\right) \frac{\pi}{H} y$$

This is a Fourier cosine series expansion of $g(y)$, thus the coefficients are:

$$-\sinh \frac{(n + \frac{1}{2})\pi L}{H} a_n = \frac{2}{H} \int_0^H g(y) \cos\left[\left(n + \frac{1}{2}\right) \frac{\pi}{H} y\right] dy$$

$$\boxed{a_n = \frac{2}{-H \sinh \frac{(n+\frac{1}{2})\pi L}{H}} \int_0^H g(y) \cos\left[\left(n + \frac{1}{2}\right) \frac{\pi}{H} y\right] dy}$$

5. c.

$$u(0,\, y) \,=\, 0 \quad \Rightarrow \quad X(0) \,=\, 0$$

$$u(L,\, y) \,=\, 0 \quad \Rightarrow \quad X(L) \,=\, 0$$

$$u(x,\, 0) \,-\, u_y\,(x,\, 0) \,=\, 0 \quad \Rightarrow \quad Y(0) \,-\, Y'(0) \,=\, 0$$

$X'' \,+\, \lambda X \,=\, 0$

$X(0) \,=\, 0$

$X(L) \,=\, 0$

$\Downarrow$

$\lambda_n \,=\, \left(\frac{n\,\pi}{L}\right)^2 \quad n \,=\, 1,\, 2, \ldots$

$X_n \,=\, \sin\, \frac{n\,\pi}{L}\, x$

$Y_n'' \,-\, \left(\frac{n\,\pi}{L}\right)^2\, Y_n \,=\, 0$

$Y_n(0) \,-\, Y_n'(0) \,=\, 0$

$Y_n \,=\, A_n\, \cosh\, \frac{n\,\pi}{L}\, y \,+\, B_n\, \sinh\, \frac{n\,\pi}{L}\, y$

$Y_n' \,=\, \frac{n\,\pi}{L}\, \left\{A_n\, \sinh\, \frac{n\,\pi}{L}\, y \,+\, B_n\, \cosh\, \frac{n\,\pi}{L}\, y\right\}$

Substitute in the boundary condition.

$$\left(A_n \,-\, \frac{n\,\pi}{L}\, B_n\right)\, \underbrace{\cosh\, 0}_{\neq 0} + \left(B_n \,-\, \frac{n\,\pi}{L}\, A_n\right) \underbrace{\sinh\, 0}_{=0} \,=\, 0$$

$$\Rightarrow \quad A_n \,=\, \frac{n\,\pi}{L}\, B_n$$

$$Y_n \,=\, B_n\left[\frac{n\,\pi}{L}\, \cosh\, \frac{n\,\pi}{L}\, y \,+\, \sinh\, \frac{n\,\pi}{L}\, y\right]$$

$$\boxed{u(x,\, y) \,=\, \sum_{n=1}^{\infty}\, b_n\, \sin\, \frac{n\,\pi}{L}\, x\, \left[\frac{n\,\pi}{L}\, \cosh\, \frac{n\,\pi}{L}\, y \,+\, \sinh\, \frac{n\,\pi}{L}\, y\right]}$$

Use the boundary condition $u(x,\, H) \,=\, f(x)$

$$\Rightarrow\, f(x) = \sum_{n=1}^{\infty}\, b_n\, \sin\, \frac{n\,\pi}{L}\, x\, \left[\frac{n\,\pi}{L}\, \cosh\, \frac{n\,\pi}{L}\, H \,+\, \sinh\, \frac{n\,\pi}{L}\, H\right]$$

This is a Fourier sine series of $f(x)$, thus the coefficients b_n are given by

$$b_n\, \left[\frac{n\,\pi}{L}\, \cosh\, \frac{n\,\pi}{L}\, H \,+\, \sinh\, \frac{n\,\pi}{L}\, H\right] \,=\, \frac{2}{L}\, \int_0^L\, f(x)\, \sin\, \frac{n\,\pi}{L}\, x\, dx$$

Solve for b_n

$$\boxed{b_n \,=\, \frac{2}{\left[\frac{n\,\pi}{L}\, \cosh\, \frac{n\,\pi}{L}\, H \,+\, \sinh\, \frac{n\,\pi}{L}\, H\right] L}\, \int_0^L\, f(x)\, \sin\, \frac{n\,\pi}{L}\, x\, dx}$$

6 a. $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ outside circle

$u(a, \theta) = \ln 2 + 4 \cos 3\theta$

$u(r, \theta) = R(r)\,\Theta(\theta)$

$r^2 R'' + rR' - \lambda R = 0$ $\qquad$ $\Theta'' + \lambda\Theta = 0$

$\Theta(0) = \Theta(2\pi)$

$\underline{\lambda = 0}\, R_0 = \alpha_0 + \beta_0 \ln r$ $\qquad$ $\Theta'(0) = \Theta'(2\pi)$

$\underline{\lambda = n^2}\, R_n = \alpha_n r^n + \beta_n r^{-n}$ $\qquad$ $\Downarrow$

Since we are solving $\qquad$ $\underline{\lambda < 0}$ trivial solution

outside the circle $\qquad$ $\underline{\lambda = 0}$ $\Theta_0(\theta) = 1$

$\ln r \to \infty$ as $r \to \infty$ $\qquad$ $\underline{\lambda > 0}$ $\lambda_n = n^2$

$r^n \to \infty$ as $r \to \infty$ $\qquad$ $\Theta_n = A_n \cos n\theta + B_n \sin n\theta$

thus $R_0 = \alpha_0$

$R_n = \beta_n r^{-n}$

$$u(r, \theta) = \underbrace{a_0\, \alpha_0 \cdot 1}_{= a_0/2} + \sum_{n=1}^{\infty} a_n \left(A_n \cos n\theta + B_n \sin n\theta\right) \beta_n r^{-n}$$

$$u(r, \theta) = a_0/2 + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right) r^{-n}$$

Use the boundary condition:

$$u(a, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n a^{-n} \cos n\theta + b_n a^{-n} \sin n\theta\right) = \ln 2 + 4 \cos 3\theta$$

$\Rightarrow b_n = 0 \quad \forall n$

$\frac{a_0}{2} = \ln 2$

$a_n a^{-n} = 4 \quad n = 3 \quad \Rightarrow \quad a_3 = 4a^3$

$a_n a^{-n} = 0 \quad n \neq 3 \quad \Rightarrow \quad a_n = 0 \quad n \neq 3$

$\Rightarrow \boxed{u(r, \theta) = \ln 2 + 4a^3 r^{-3} \cos 3\theta}$

6 b. The only difference between this problem and the previous one is the boundary condition

$$u(a,\,\theta) \;=\; f(\theta) \;=\; \frac{a_0}{2} \;+\; \sum_{n=1}^{\infty}\left(a_n\, a^{-n}\, \cos\, n\,\theta \;+\; b_n\, a^{-n}\, \sin\, n\,\theta\right)$$

$\Rightarrow\; a_0,\;\; a_n\, a^{-n},\;\; b_n\, a^{-n}$ are coefficients of Fourier series of f

$$a_0 \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\, d\,\theta$$

$$a_n\, a^{-n} \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\, \cos\, n\,\theta\, d\,\theta$$

$$b_n\, a^{-n} \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\, \sin\, n\,\theta\, d\,\theta$$

Divide the last two equations by a^{-n} to get the coefficients a_n and b_n.

7 a. $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$

$u_\theta(r, 0) = 0$

$u(r, \pi/2) = 0$

$u(1, \theta) = f(\theta)$

$r^2 R'' + r R' - \lambda R = 0$

$R_n = c_n r^{2n-1} + D_n r^{2n-1}$

boundedness implies $R_n = c_n r^{2n-1}$

$\Theta'' + \lambda \Theta = 0$ no periodicity !!

$\Theta'(0) = 0$

$\Theta'(\pi/2) = 0$

If $\underline{\lambda < 0}$ trivial

$\underline{\lambda = 0}\, \Theta_0 = A_0 \theta + B_0$

$\Theta_0' = A_0 \quad \Theta_0'(0) = 0 \quad \Rightarrow \quad A_0 = 0$

$\Theta_0(\pi/2) = 0 \quad \Rightarrow \quad B_0 = 0$

trivial

$\underline{\lambda > 0}\, \Theta = A \cos \sqrt{\lambda}\theta + B \sin \sqrt{\lambda}\theta$

$\Theta' = -\sqrt{\lambda} A \sin \sqrt{\lambda}\theta + B\sqrt{\lambda} \cos \sqrt{\lambda}\theta$

$\Theta'(0) = 0 \Rightarrow \quad B = 0$

$\Theta(\pi/2) = 0 \Rightarrow A \cos \sqrt{\lambda}\pi/2 = 0$

$\sqrt{\lambda}\pi/2 = \left(n - \frac{1}{2}\right)\pi \quad n = 1, 2, \cdots$

$\sqrt{\lambda} = 2\left(n - \frac{1}{2}\right) = 2n - 1$

$\boxed{\lambda_n = (2n - 1)^2}$

$\Theta_n = \cos(2n - 1)\,\theta \quad , \quad n = 1, 2, \cdots$

Therefore the solution is

$$u = \sum_{n=1}^{\infty} a_n r^{2n-1} \cos(2n - 1)\theta$$

Use the boundary condition

$$u(1,\,\theta) \,=\, \sum_{n=1}^{\infty} a_n \, \cos{(2n\,-\,1)\,\theta} \,=\, f(\theta)$$

This is a Fourier cosine series of $f(\theta)$, thus the coefficients are given by

$$a_n \,=\, \frac{2}{\pi/2} \int_0^{\pi/2} f(\theta) \, \cos{(2n\,-\,1)\,\theta}\, d\,\theta$$

Remark: Since there is no constant term in this Fourier cosine series, we should have

$$a_0 \,=\, \frac{2}{\frac{\pi}{2}} \int_0^{\pi/2} f(\theta)\, d\theta \,=\, 0$$

That means that the boundary condition on the curved part of the domain is not arbitrary but must satisfy

$$\int_0^{\pi/2} f(\theta)\, d\theta \,=\, 0$$

7 b. $u_\theta(r, 0) = 0$

$u_\theta(r, \pi/2) = 0$

$u_r(1, \theta) = g(\theta)$

Use 7 a to get the 2 ODEs

$$\Theta'' + \lambda\Theta = 0 \qquad\qquad r^2 R'' + r R' - \lambda R = 0$$

$$\Theta'(0) = \Theta'(\pi/2) = 0$$

$$\Downarrow$$

$$\lambda_n = \left(\frac{n\pi}{\frac{\pi}{2}}\right)^2 = (2n)^2 \quad n = 0, 1, 2, \ldots$$

$$\Theta_n = \cos 2n\theta, \qquad n = 0, 1, 2, \ldots$$

Now substitute the eigenvalues in the R equation
$r^2 R'' + r R' - (2n)^2 R = 0$
The solution is

$$R_0 = C_0 \ln r + D_0, \qquad n = 0$$

$$R_n = C_n r^{-2n} + D_n r^{2n}, \qquad n = 1, 2, \ldots$$

Since $\ln r$ and r^{-2n} blow up as $r \to 0$ we have $C_0 = C_n = 0$. Thus

$$\boxed{u(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n r^{2n} \cos 2n\theta}$$

Apply the inhomogeneous boundary condition

$$u_r(r,\theta) = \sum_{n=1}^{\infty} 2n\, a_n r^{2n-1} \cos 2n\theta$$

And at $r = 1$

$$u_r(1,\theta) = \sum_{n=1}^{\infty} 2n\, a_n \cos 2n\theta = g(\theta)$$

This is a Fourier cosine series for $g(\theta)$ and thus

$$2n\, a_n = \frac{\int_0^{\pi/2} g(\theta) \cos 2n\theta\, d\theta}{\int_0^{\pi/2} \cos^2 2n\theta\, d\theta}$$

$$\boxed{a_n = \frac{\int_0^{\pi/2} g(\theta) \cos 2n\theta\, d\theta}{2n \int_0^{\pi/2} \cos^2 2n\theta\, d\theta} \qquad n = 1, 2, \ldots}$$

Note: a_0 is still arbitrary. Thus the solution is <u>not</u> unique. Physically we require $\displaystyle\int_0^{\pi/2} g(\theta)\, d\theta = 0$ which is to say that $a_0 = 0$.

7 c. $u(r, 0) = 0$

$u(r, \pi/2) = 0$

$u_r(1, \theta) = 1$

Use 7 a to get the 2 ODEs

$\Theta'' + \lambda \Theta = 0 \qquad\qquad r^2 R'' + r R' - \lambda R = 0$

$\Theta(0) = \Theta(\pi/2) = 0$

$\Downarrow$

$\lambda_n = \left(\frac{n\pi}{\frac{\pi}{2}}\right)^2 = (2n)^2 \quad n = 1, 2, \ldots$

$\Theta_n = \sin 2n\theta, \qquad n = 1, 2, \ldots$

Now substitute the eigenvalues in the R equation
$r^2 R'' + r R' - (2n)^2 R = 0$
The solution is

$$R_n = C_n r^{-2n} + D_n r^{2n}, \qquad n = 1, 2, \ldots$$

Since r^{-2n} blow up as $r \to 0$ we have $C_n = 0$. Thus

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} a_n r^{2n} \sin 2n\theta}$$

Apply the inhomogeneous boundary condition

$$u_r(r,\theta) = \sum_{n=1}^{\infty} 2n\, a_n r^{2n-1} \sin 2n\theta$$

And at $r = 1$

$$u_r(1,\theta) = \sum_{n=1}^{\infty} 2n\, a_n \sin 2n\theta = 1$$

This is a Fourier sine series for the constant function 1 and thus

$$2n\, a_n = \frac{\int_0^{\pi/2} 1 \cdot \sin 2n\theta\, d\theta}{\int_0^{\pi/2} \sin^2 2n\theta\, d\theta}$$

$$a_n = \frac{\int_0^{\pi/2} 1 \cdot \sin 2n\theta\, d\theta}{2n \int_0^{\pi/2} \sin^2 2n\theta\, d\theta} = \frac{\frac{1-(-1)^n}{2n}}{2n\frac{\pi}{2}} = \frac{1-(-1)^n}{n^2\pi}$$

$$\boxed{a_n = \frac{1-(-1)^n)}{n^2\pi}}$$

8a.
$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$
$$u(a, \theta) = f(\theta)$$
$$u(b, \theta) = g(\theta)$$

$$r^2 R'' + rR' - \lambda R = 0 \qquad \Theta'' + \lambda\Theta = 0$$
$$\Theta(0) = \Theta(2\pi)$$
$$\Theta'(0) = \Theta'(2\pi)$$

The eigenvalues and eigenfunctions can be found in the summary of chapter 4

$$\lambda_0 = 0 \qquad \Theta_0 = 1 \qquad \text{for } n = 0$$
$$\lambda_n = n^2 \qquad \Theta_n = \cos n\theta \text{ and } \sin n\theta \qquad \text{for } n = 1, 2, \ldots$$

Use these eigenvalues in the R equation and we get the following solutions:

$$R_0 = A_0 + B_0 \ln r \qquad n = 0$$
$$R_n = A_n r^n + B_n r^{-n} \qquad n = 1, 2, \ldots$$

Since $r = 0$ is outside the domain and r is finite, we have no reason to throw away any of the 4 parameters A_0, A_n, B_0, B_n.

Thus the solution

$$u(r, \theta) = \underbrace{(A_0 + B_0 \ln r)}_{R_0} \cdot \underbrace{1}_{\Theta_0} \cdot a_0 + \sum_{n=1}^{\infty} \underbrace{(A_n r^n + B_n r^{-n})}_{R_n} \underbrace{(a_n \cos n\theta + b_n \sin n\theta)}_{\Theta_n}$$

Use the 2 inhomogeneous boundary conditions

$$f(\theta) = u(a, \theta) = \underbrace{A_0 a_0 + B_0 a_0 \ln a}_{\alpha_0} + \sum_{n=1}^{\infty} \underbrace{(A_n a^n + B_n a^{-n}) a_n}_{\alpha_n} \cos n\theta$$

$$+ \sum_{n=1}^{\infty} \underbrace{(A_n a^n + B_n a^{-n}) b_n}_{\beta_n} \sin n\theta$$

$$g(\theta) = u(b, \theta) = \underbrace{A_0 a_0 + B_0 a_0 \ln b}_{\gamma_0} + \sum_{n=1}^{\infty} \underbrace{(A_n b^n + B_n b^{-n}) a_n}_{\gamma_n} \cos n\theta$$

$$+\sum_{n=1}^{\infty} \underbrace{(A_n\, b^n \,+\, B_n\, b^{-n})\, b_n}_{\delta_n} \sin\, n\,\theta$$

These are Fourier series of $f(\theta)$ and $g(\theta)$ thus the coefficients α_0, α_n, β_n for f and the coefficients γ_0, γ_n, δ_n for g can be written as follows

$$\begin{aligned}
\alpha_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta \\
\alpha_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\, \cos n\theta\, d\theta \\
\beta_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\, \sin n\theta\, d\theta \\
\gamma_0 &= \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\, d\theta \\
\gamma_n &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\, \cos n\theta\, d\theta \\
\delta_n &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\, \sin n\theta\, d\theta
\end{aligned}$$

On the other hand these coefficients are related to the unknowns $A_0, a_0, B_0, b_0, A_n, a_n, B_n$ and b_n via the three systems of 2 equations each

$$\left.\begin{aligned}
\alpha_0 &= A_0\, a_0 \,+\, B_0\, a_0\, \ln\, a \\
\gamma_0 &= A_0\, a_0 \,+\, B_0\, a_0\, \ln\, b
\end{aligned}\right\}\quad \text{solve for } A_0\, a_0\, ,\; B_0\, a_0$$

$$\left.\begin{aligned}
\alpha_n &= (A_n\, a^n \,+\, B_n\, a^{-n})\, a_n \\
\gamma_n &= (A_n\, b^n \,+\, B_n\, b^{-n})\, a_n
\end{aligned}\right\}\quad \text{solve for } A_n\, a_n, B_n\, a_n$$

$$\left.\begin{aligned}
\beta_n &= (A_n\, a^n \,+\, B_n\, a^{-n})\, b_n \\
\delta_n &= (A_n\, b^n \,+\, B_n\, b^{-n})\, b_n
\end{aligned}\right\}\quad \text{solve for } A_n\, b_n, B_n\, b_n$$

Notice that we only need the products $A_0 a_0$, $B_0 b_0$,$A_n a_n$, $B_n a_n$, $A_n b_n$, and $B_n b_n$.

$$B_0 a_0 \,=\, \frac{\gamma_0 \,-\, \alpha_0}{\ln b \,-\, \ln a}$$

$$A_0 a_0 \;=\; \frac{\alpha_0 \,\ln b \;-\; \gamma_0 \,\ln a}{\ln b \;-\; \ln a}$$

$$B_n a_n \;=\; \frac{\alpha_n b^n \;-\; \gamma_n a^n}{b^n a^{-n} \;-\; a^n b^{-n}}$$

$$A_n a_n \;=\; \frac{\gamma_n b^n \;-\; \alpha_n a^n}{b^{2n} \;-\; a^{2n}}$$

In a similar fashion

$$B_n b_n \;=\; \frac{\beta_n b^n \;-\; \delta_n a^n}{b^n a^{-n} \;-\; a^n b^{-n}}$$

$$A_n b_n \;=\; \frac{\delta_n b^n \;-\; \beta_n a^n}{b^{2n} \;-\; a^{2n}}$$

8. b Similar to 8.a

$$u(r,\,\theta) \,=\, (A_0 \,+\, B_0 \,\ln\, r)\,a_0 \,+\, \sum_{n=1}^{\infty} (A_n\,r^n \,+\, B_n\,r^{-n})\,(a_n\,\cos\,n\,\theta \,+\, b_n\,\sin\,n\,\theta)$$

To use the boundary conditions:

$$u_r(a,\,\theta) \,=\, f(\theta)$$

$$u_r(b,\,\theta) \,=\, g(\theta)$$

We need to differentiate u with respect to r

$$u_r(r,\,\theta) \,=\, \frac{B_0}{r}\,a_0 \,+\, \sum_{n=1}^{\infty} (n\,A_n\,r^{n-1} \,-\, n\,B_n\,r^{-n-1})\,(a_n\,\cos\,n\,\theta \,+\, b_n\,\sin\,n\,\theta)$$

Substitute $r \,=\, a$

$$u_r(a,\,\theta) \,=\, \frac{B_0}{a}\,a_0 \,+\, \sum_{n=1}^{\infty} (n\,A_n\,a^{n-1} \,-\, n\,B_n\,a^{-n-1})\,(a_n\,\cos\,n\,\theta \,+\, b_n\,\sin\,n\,\theta)$$

This is a Fourier series expansion of $f(\theta)$ thus the coefficients are

$$\frac{B_0}{a}a_0 \,=\, \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta \,\equiv\, \alpha_0$$

$$(nA_na^{n-1} \,-\, nB_na^{-n-1})a_n \,=\, \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,\cos\,n\theta\,d\theta \,\equiv\, \alpha_n$$

$$(nA_na^{n-1} \,-\, nB_na^{-n-1})b_n \,=\, \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,\sin\,n\theta\,d\theta \,\equiv\, \beta_n$$

Now substitute $r \,=\, b$

$$u_r(b,\,\theta) \,=\, \frac{B_0}{b}\,a_0 \,+\, \sum_{n=1}^{\infty} (n\,A_n\,b^{n-1} \,-\, n\,B_n\,b^{-n-1})\,(a_n\,\cos\,n\,\theta \,+\, b_n\,\sin\,n\,\theta)$$

This is a Fourier series expansion of $g(\theta)$ thus the coefficients are

$$\frac{B_0}{b}a_0 \,=\, \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,d\theta \,\equiv\, \gamma_0$$

$$(nA_nb^{n-1} \,-\, nB_nb^{-n-1})a_n \,=\, \frac{1}{\pi}\int_0^{2\pi} g(\theta)\,\cos\,n\theta\,d\theta \,\equiv\, \gamma_n$$

$$(nA_n b^{n-1} - nB_n b^{-n-1})b_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin n\theta\, d\theta \equiv \delta_n$$

Solve for $A_n a_n$, $B_n a_n$:

$$(nA_n a^{n-1} - nB_n a^{-n-1})a_n = \alpha_n$$

$$(nA_n b^{n-1} - nB_n b^{-n-1})a_n = \gamma_n$$

We have

$$A_n a_n = \frac{\alpha_n\, b^{-n-1} - \gamma_n\, a^{-n-1}}{n(a^{n-1}\, b^{-n-1} - b^{n-1}\, a^{-n-1})}$$

$$B_n a_n = \frac{\alpha_n\, b^{n-1} - \gamma_n\, a^{n-1}}{n(a^{n-1}\, b^{-n-1} - b^{n-1}\, a^{-n-1})}$$

Solve for $A_n b_n$, $B_n b_n$:

$$(nA_n a^{n-1} - nB_n a^{-n-1})b_n = \beta_n$$

$$(nA_n b^{n-1} - nB_n b^{-n-1})b_n = \delta_n$$

We have

$$A_n b_n = \frac{\beta_n\, b^{-n-1} - \delta_n\, a^{-n-1}}{n(a^{n-1}\, b^{-n-1} - b^{n-1}\, a^{-n-1})}$$

$$B_n b_n = \frac{\beta_n\, b^{n-1} - \delta_n\, a^{n-1}}{n(a^{n-1}\, b^{-n-1} - b^{n-1}\, a^{-n-1})}$$

There are two equations for $B_0 a_0$:

$$B_0 a_0 = b\gamma_0$$

$$B_0 a_0 = a\alpha_0$$

This means that f and g are not independent, but

$$a\alpha_0 = b\gamma_0$$

which means that

$$a\int_0^{2\pi} f(\theta)\, d\theta = b\int_0^{2\pi} g(\theta)\, d\theta$$

Note also that there is <u>no</u> condition on $A_0 a_0$.

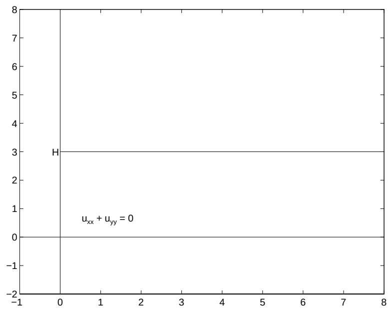


Figure 57: Sketch of domain

9.

$u_y\,(x,\,0) \,=\, 0$

$u_y\,(x,\,H) \,=\, 0$

$u\,(0,\,y) \,=\, f(y)$

$X'' \,-\, \lambda\, X \,=\, 0$	$Y'' \,+\, \lambda\, Y \,=\, 0$	
solution should	$Y'(0) \,=\, 0$	
be bounded	$Y'(H) \,=\, 0$	
when $x \,\to\, \infty$		copy from table in Chapter 4 summary
	$\lambda_n \,=\, \left(\frac{n\,\pi}{H}\right)^2$	$n \,=\, 0,\, 1,\, 2,\, \cdots$
	$Y_n \,=\, \cos\, \frac{n\,\pi}{H}\, y$	

$X_n'' \,-\, \left(\frac{n\,\pi}{H}\right)^2\, X_n \,=\, 0 \qquad n \,=\, 1,\, 2,\, \cdots$

$X_n \,=\, A_n\, e^{\frac{n\,\pi}{H}\, x} \,+\, B_n\, e^{-\,\frac{n\,\pi}{H}\, x}$

to get bounded solution $A_n = 0$

For $n \,=\, 0$

$X_0'' \,=\, 0$

$X_0 \,=\, A_0 x \,+\, B_0$
for boundedness $A_0 = 0$

$$u \,=\, B_0 \,\cdot\, 1 \,+\, \sum_{n=1}^{\infty} B_n\, e^{-\,\frac{n\,\pi}{H}\, x}\, \cos\, \frac{n\,\pi}{H}\, y$$

$$u(0,\,y) \,=\, f(y) \,=\, B_0 \,+\, \sum_{n=1}^{\infty} B_n\, \cos\, \frac{n\,\pi}{H}\, y$$

Fourier cosine series of $f(y)$.

10. $\quad u_t = u_{xx} + q(x,\, t) \qquad 0 < x < L$

subject to BC $\quad u(0,\, t) = u(L,\, t) = 0$

Assume: $q(x,\, t)$ piecewise smooth for each positive t.

u and u_x continuous

u_{xx} and u_t piecewise smooth.

Thus,

$$u(x,\, t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi}{L} x$$

(a). Write the ODE satisfied by $b_n(t)$, and

(b). Solve this heat equation.

<u>STEPS:</u>

1. Compute $q_n(t)$, the known heat source coefficient
2. Plug u and q series expansions into PDE.
3. Solve for $b_n(t)$ - the homogeneous and particular solutions, $b_n^H(t)$ and $b_n^P(t)$
4. Apply initial condition, $b_n(0)$, to find coefficient A_n in the $b_n(t)$ solution. Assume $u(x,\, 0) = f(x)$

1.

$$q(x,\, t) = \sum_{n=1}^{\infty} q_n(t) \sin \frac{n\pi}{L} x$$

$$q_n(t) = \frac{2}{L} \int_0^L q(x,\, t) \sin \frac{n\pi}{L} x \, dx$$

2.

$$u_t = \sum_{n=1}^{\infty} b_n'(t) \sin \frac{n\pi}{L} x$$

$$u_{xx} = \sum_{n=1}^{\infty} b_n(t) \left[-\left(\frac{n\pi}{L} \right)^2 \right] \sin \frac{n\pi}{L} x$$

$$\sum_{n=1}^{\infty} b_n'(t) \sin \frac{n\pi}{L} x = \sum_{n=1}^{\infty} b_n'(t) \left[-\left(\frac{n\pi}{L}\right)^2 \right] \sin \frac{n\pi}{L} x + \sum_{n=1}^{\infty} q_n(t) \sin$$

We have a Fourier Sine series on left and Fourier Sine series on right, so the coefficients must be the same; i.e.,

(a) $\boxed{b_n'(t) = -\left(\frac{n\pi}{L}\right)^2 b_n(t) + q_n(t)}$ $\Rightarrow$ A first order ODE for $b_n(t)$.

III. Solve $b_n'(t) = -\left(\frac{n\pi}{L}\right)^2 b_n(t) + q_n(t)$

Solution Form: $b_n(t) = A_n\, b_n^H(t) + b_n^P(t)$

Homogeneous Solution: $b_n^H(t) = e^{-\left(\frac{n\pi}{L}\right)^2 t}$

Particular Solution: $b_n^P(t) = e^{\left(\frac{n\pi}{L}\right)^2 t} \int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 \tau} q_n(\tau)\, d\tau$

$$\boxed{b_n(t) = A_n\, e^{-\left(\frac{n\pi}{L}\right)^2 t} + e^{\left(\frac{n\pi}{L}\right)^2 t} \int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 \tau} q_n(\tau)\, d\tau}$$

(Step IV is an extra step, not required in homework problem.)

<u>IV</u>. Find A_n from initial condition. $u(x, 0) = f(x) = \sum_{n=1}^{\infty} b_n(0) \sin \frac{n\pi}{L} x$

$$b_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x\, dx$$

$$b_n(0) = A_n + e^0 \int_0^0 e^{-\left(\frac{n\pi}{L}\right)^2 \tau} q_n(\tau)\, dt$$

$$b_n(0) = A_n + 1 \cdot 0 = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x\, dx$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x\, dx$$

$$b_n(t) = \frac{2}{L} \int_0^L \left(f(x) \sin \frac{n\pi}{L} x\, dx \right) \left(e^{-\left(\frac{n\pi}{L}\right)^2 t} \right) + e^{\left(\frac{n\pi}{L}\right)^2 t} \int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 \tau} q_n(\tau)\, d\tau$$

Plug this into $u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi}{L} x$

11. $u_t = k\, u_{xx} + \underbrace{e^{-2t} \cos \frac{3\,\pi}{L}}_{q\,(x,\,t)}\, x$

$u_x\,(0,\,t) = 0$

$u_x\,(L,\,t) = 0$

$u(x,\,0) = f(x)$

The boundary conditions imply

$$u(x,\,t) = \sum_{n=0}^{\infty} b_n(t) \cos \frac{n\,\pi}{L}\, x$$

Let $\quad q(x,\,t) = \sum_{n=0}^{\infty} q_n(t) \cos \frac{n\,\pi}{L}\, x \quad \Rightarrow \quad q_0(t) = e^{-t}$

$q_3(t) = e^{-2t}$ the rest are zero !

Thus

$$\dot{b}_n = -k \left(\frac{n\,\pi}{L}\right)^2 b_n + q_n \qquad n = 0,\, 1,\, \cdots$$

$\underline{n = 0} \qquad \dot{b}_0 = q_0 = e^{-t} \quad \Rightarrow \quad \underline{b_0 = -\,e^{-t}}$

$$\left.\begin{aligned} \dot{b}_1 + k \left(\tfrac{n\,\pi}{L}\right)^2 b_1 &= q_1 = 0 \\ \dot{b}_2 + k \left(\tfrac{2n\,\pi}{L}\right)^2 b_2 &= 0 \end{aligned}\right\} \text{homogeneous}$$

$$\dot{b}_3 + k \left(\frac{3n\,\pi}{L}\right)^2 b_3 = e^{-2t}$$

rest are homogeneous.

One can solve each equation to obtain all b_n.

$\dot{b}_n + k \left(\frac{n\,\pi}{L}\right)^2 b_n = 0 \quad \Rightarrow \quad \boxed{b_n = C_n e^{-k\left(\frac{n\,\pi}{L}\right)^2 t} \quad n = 1,\, 2,\, 4,\, 5,\, \cdots \quad \text{note: } n \neq 3}$

$\dot{b}_3 + k \left(\frac{3\,\pi}{L}\right)^2 b_3 = e^{-2t}$ Solution of homogeneous is

$$b_3 = C_3\, e^{-k\left(\frac{3\,\pi}{L}\right)^2 t}$$

For particular solution try $\quad b_3 = C\, e^{-2t}$

$$\Rightarrow \quad -2C\,e^{-2t} + k\left(\frac{3\,\pi}{L}\right)^2 = C\,e^{-2t} = e^{-2t}$$

$$\left[-2 + k\left(\frac{3\,\pi}{L}\right)^2\right] C = 1$$

$$C = \frac{1}{k\left(\frac{3\,\pi}{L}\right)^2 - 2}$$

denominator is not zero as assumed in the problem.

$$\Rightarrow \quad \boxed{b_3 = C_3\,e^{k\left(\frac{3\,\pi}{L}\right)^2 t} + \frac{1}{k\left(\frac{3\,\pi}{L}\right)^2 - 2}\,e^{-2t}}$$

12. $u_{tt} - c^2 u_{xx} = 0 \qquad 0 < x < L$

$$u(0,t) = u(L,\, t) = 0$$

$$u(x,\, 0) = f(x)$$

$$u_t(x,\, 0) = g(x)$$

$$XT'' - c^2 X'' T = 0$$

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$$

$$X'' + \lambda X = 0 \qquad\qquad T'' + \lambda c^2 T = 0$$

$$X(0) = X(L) = 0$$

$$X_n = \sin \frac{n\pi}{L} x \qquad\qquad T_n'' + \left(\frac{n\pi}{L}\right)^2 c^2 T_n = 0$$

$$n = 1,\, 2,\, \cdots$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad\qquad T_n = \alpha_n \cos \frac{n\pi c}{L} t + \beta_n \sin \frac{n\pi c}{L} t$$

$$\boxed{u(x,\, t) = \sum_{n=1}^{\infty} \left\{ \alpha_n \cos \frac{n\pi c}{L} t + \beta_n \sin \frac{n\pi c}{L} t \right\} \sin \frac{n\pi}{L} x}$$

$$u(x,\, 0) = f(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi}{L} x$$

$$u_t(x,\, 0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} \beta_n \sin \frac{n\pi}{L} x$$

$$\boxed{\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x\, dx}$$

$$\frac{n\pi c}{L} \beta_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x\, dx$$

$$\boxed{\beta_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi}{L} x\, dx}$$

13. $u_t = 2u_{xx}$

$u(0, t) = 0$

$u_x(L, t) = 0$

$u(x, 0) = \sin \frac{3}{2} \frac{\pi}{L} x$

$u = XT$

$X\dot{T} = 2X''T$

$$\frac{\dot{T}}{2T} = \frac{X''}{X} = -\lambda \qquad X'' + \lambda X = 0 \qquad \dot{T} + 2\lambda T = 0$$

$$X(0) = 0$$

$$X'(L) = 0$$

$$X_n = \sin\left(n + \frac{1}{2}\right)\frac{\pi}{L}x \qquad n = 0, 1, \cdots$$

$$\lambda_n = \left[\left(n + \frac{1}{2}\right)\frac{\pi}{L}\right]^2$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \, e^{-2\left[\left((n+\frac{1}{2}\right)\frac{\pi}{L}\right]^2 t} \sin\left(n + \frac{1}{2}\right)\frac{\pi}{L}x$$

At $t = 0$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(n + \frac{1}{2}\right)\frac{\pi}{L}x$$

But also
$u(x, 0) = \sin \frac{3}{2} \frac{\pi}{L} x$
Therefore

$$a_1 = 1, \quad a_n = 0 \quad n > 1$$

$$\boxed{u(x,t) = e^{-2\left(\frac{3}{2}\frac{\pi}{L}\right)^2 t} \sin \frac{3}{2}\frac{\pi}{L}x}$$

14. $u_t = k\left[\frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta}\right]$

$u_r(a, \theta, t) = 0$ inside a disk

$u(r, \theta, 0) = f(r, \theta)$

$\Theta T' R = kT \left[\Theta \frac{1}{r}(r R')' + \frac{1}{r^2} R \Theta''\right]$

$$\frac{T'}{kT} = \frac{\frac{1}{r}(rR')'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda$$

$T' + \lambda k T = 0$ $\qquad \frac{\frac{1}{r}(rR')'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda$ multiply through by r^2

$$\frac{r(rR')'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta} = \mu$$

$\Theta'' + \mu \Theta = 0;$ $\qquad r(rR')' + \lambda r^2 R - \mu R = 0$

$\Theta(0) = \Theta(2\pi)$ $\qquad |R(0)| < \infty$

$\Theta'(0) = \Theta'(2\pi)$ $\qquad R'(a) = 0$

$\Downarrow$ $\qquad \downarrow$

$\mu_n = n^2$ $\qquad R = J_n(\sqrt{\lambda} r)$

$n = 1, 2, \cdots$

$\Theta_n = \begin{cases} \sin n\theta \\ \cos n\theta \end{cases}$ $\qquad \boxed{J_n'(\sqrt{\lambda} a) = 0 \quad \text{gives } \lambda_{nm}}$

$\mu_0 = 0 \qquad \Theta_0 = 1$

$T_{nm}' + \lambda_{nm} k T_{nm} = 0 \rightarrow T_{nm} = e^{-\lambda_{nm} k t}$

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \underbrace{\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)\right]}_{\Theta_n} \underbrace{J_n\left(\sqrt{\lambda_{nm}} r\right)}_{R_{nm}} \underbrace{e^{-k\lambda_{nm} t}}_{T_{nm}}$$

$$f(r, \theta) = \sum_{m=1}^{\infty}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)\right] J_n\left(\sqrt{\lambda_{nm}} r\right)$$

Fourier-Bessel expansion of f.

See (7.5 later)

6 Sturm-Liouville Eigenvalue Problem

6.1 Introduction

Problems

1. a. Show that the following is a regular Sturm-Liouville problem

$$X''(x) + \lambda X(x) = 0,$$

$$X(0) = 0,$$
$$X'(L) = 0.$$

b. Find the eigenpairs λ_n, X_n directly.

c. Show that these pairs satisfy the results of the theorem.

2. Prove (6.1.28) - (6.1.30).

3. a. Is the following a regular Sturm-Liouville problem?

$$X''(x) + \lambda X(x) = 0,$$

$$X(0) = X(L),$$
$$X'(0) = X'(L).$$

Why or why not?
   b. Find the eigenpairs λ_n, X_n directly.
   c. Do they satisfy the results of the theorem? Why or why not?

4. Solve the regular Sturm-Liouville problem

$$X''(x) + aX(x) + \lambda X(x) = 0, \qquad a > 0,$$

$$X(0) = X(L) = 0.$$

For what range of values of a is λ negative?

5. Solve the ODE

$$X''(x) + 2\alpha X(x) + \lambda X(x) = 0, \qquad \alpha > 1,$$

$$X(0) = X'(1) = 0.$$

6. Consider the following Sturm-Liouville eigenvalue problem

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) + \lambda\frac{1}{x}u = 0, \qquad 1 < x < 2,$$

with boundary conditions

$$u(1) = u(2) = 0.$$

Determine the sign of all the eigenvalues of this problem (you don't have to explicitly determine the eigenvalues). In particular, is zero an eigenvalue of this problem?

7. Consider the following model approximating the motion of a string whose density (along the string) is proportional to $(1+x)^{-2}$,

$$(1+x)^{-2}u_{tt} - u_{xx} = 0, \qquad 0 < x < 1, \quad t > 0$$

subject to the following initial conditions

$$u(x,0) = f(x),$$

$$u_t(x,0) = 0,$$

and boundary conditions

$$u(0,t) = u(L,t) = 0.$$

a. Show that the ODE for X resulting from separation of variables is

$$X'' + \frac{\lambda}{(1+x)^2}X = 0.$$

b. Obtain the boundary conditions and solve.
Hint: Try $X = (1+x)^a$.

1. a $X'' + \lambda X = 0$

$X(0) = X'(L) = 0$

is regular:

$p = 1 \qquad q = 0 \qquad \sigma = 1$

p continuous & positive & differentiable

q continuous & nonnegative

σ continuous & positive

$\beta_1 = 1 \qquad \beta_2 = 0 \qquad \beta_3 = 0 \qquad \beta_4 = 1$

b. See chapter 4

$$\lambda_n = \left[\left(n - \frac{1}{2}\right)\frac{\pi}{L}\right]^2 \qquad n = 1, 2, \cdots$$

$$X_n = \sin\frac{\left(n - \frac{1}{2}\right)\pi}{L}x$$

c. Infinitely many eigenvalues

λ_1 is smallest

no largest

X_n are orthogonal

one eigenvalue for each eigenfunctions

and so on

2. The eigenfunctions $X_n(x)$ and the eigenvalues are λ_n. Use these λ_n in (6.1.24)
$\dot{T}_n + \lambda_n T_n = 0$

$$\Rightarrow \quad T_n(t) = C_n e^{-\lambda_n T}$$

$$\Rightarrow u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} X_n(x)$$

at $t = 0$ $\quad u(x, 0) = \sum_{n=1}^{\infty} a_n X_n(x), \quad a_n \quad$ are $\quad T_n(0) \quad$ in $\quad$ (6.1.28)

To find a_n we use the Fourier series expansion of $f(x) = u(x, 0)$

$$\Rightarrow a_n \frac{\int_0^L f(x) X_n(x) c(x) \rho(x) dx}{\int_0^L X_n^2(x) \underbrace{c(x) \rho(x)}_{\text{weight function}} dx}$$

If $a_1 \neq 0$ clearly λ_1 is the smallest and thus $a_1 e^{-\lambda_1 t} X_1(x)$ dies the slowest.

3a. No, because the boundary conditions are <u>not</u> of the form

$$\beta_1 X(0) + \beta_2 X'(0) = 0$$

$$\beta_3 X(L) + \beta_4 X'(L) = 0$$

b. eigenpairs found in chapter 4

$$\lambda_0 = 0 \qquad X_0 = 1$$

$$\lambda_n = \left(\frac{2n\pi}{L}\right)^2 \qquad X_n = \begin{cases} \sin & \frac{2n\pi}{L}x \\ \cos & \frac{2n\pi}{L}x \end{cases}$$

c. No, because we have more than one eigenfunction for some eigenvalues.

4.

$$X'' + aX + \lambda X = 0 \qquad a > 0$$

$$X(0) = X(L) = 0$$

$$X = e^{\mu x}$$

$$\mu^2 + a + \lambda = 0$$

$$\mu = \pm\sqrt{-a - \lambda}$$

If $a + \lambda > 0 \qquad (\lambda > -a)$

$$X = A cos\sqrt{a + \lambda}\,x + B \sin\sqrt{a + \lambda}\,x$$

$$\Rightarrow \qquad X = B \sin\sqrt{a + \lambda}\,L = 0$$

$$\sqrt{a + \lambda}\,L = n\pi \qquad n = 1, 2, \cdots$$

$$a + \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

$\boxed{\lambda_n = -a + \left(\frac{n\pi}{L}\right)^2} \qquad n = 1, 2, \cdots$

(show that if $a + \lambda \leq 0$ the solution is trivial)

$$\lambda < 0 \qquad \text{if} \qquad -a + \left(\frac{\pi}{L}\right)^2 < 0$$

$$\boxed{a > (\pi/L)^2}$$

5.

$$X'' + 2\,\alpha\,X + \lambda\,X = 0 \qquad \alpha > 1$$

$$X(0) = X'(1) = 0$$

$$\mu^2 + 2\alpha + \lambda = 0$$

$$\mu = \pm\sqrt{-\lambda - 2\alpha}$$

If $\lambda + 2\alpha > 0 \qquad (\lambda > -2\alpha)$

$$X = A\cos\sqrt{\lambda + 2\alpha}\,x + B\sin\sqrt{\lambda + 2\alpha}\,x$$

$$X(0) = 0 \Rightarrow A = 0$$

$$X'(1) = B\sqrt{\lambda + 2\alpha}\cos\sqrt{\lambda + 2\alpha} = 0$$

$$\sqrt{\lambda + 2\alpha} = \frac{\pi}{2} + n\,\pi \qquad n = 0, 1, 2, \cdots$$

$$\left.\begin{aligned} \lambda_n &= -2\alpha + \left[\left(n + \frac{1}{2}\right)\pi\right]^2 \\ X_n &= \sin\left(n + \frac{1}{2}\right)\pi\,x \end{aligned}\right\} n = 0, 1, 2, \cdots$$

If $\alpha > 1$, λ could be negative if $\alpha > \frac{\pi^2}{8}$

If $1 < \alpha < \frac{\pi^2}{8}$ then all λ are > 0.

6.
$$(x\,u')' + \lambda \frac{1}{x} u = 0 \qquad 1 < x < 2$$

$$u(1) = u(2) = 0$$

Use Rayleigh quotient $\qquad p = x, \qquad q = 0, \qquad \sigma = \dfrac{1}{x}$

$$\lambda = \frac{-x\,u\,u'\,|_1^2 + \int_1^2 x\,(u')^2\,dx}{\int_1^2 \frac{1}{x} u^2\,dx} = \frac{\int_1^2 x\,(u')^2\,dx}{\int_1^2 \frac{1}{x} u^2\,dx}$$

denominator is positive

numerator could be zero if u = constant. $\Rightarrow \lambda \geq 0$

Is that $(\lambda = 0)$ a possibility?

$$\lambda = 0 \Rightarrow (xu')' = 0$$

Integrate

$$xu' = c = \text{constant}$$

$$u' = \tfrac{c}{x}$$

Integrate again

$$u = c \log x + d$$

$$u(1) = c \cdot 0 + d = 0 \Rightarrow d = 0$$

$$u(2) = 0 \Rightarrow c \log 2 = 0 \Rightarrow c = 0$$

$\Rightarrow \lambda = 0$ is <u>not</u> an eigenvalue.

7. $(1+x)^{-2} u_{tt} - u_{xx} = 0, \qquad 0 < x < 1$

$u(x, 0) = f(x)$

$u_t(x, 0) = 0$

$u(0, t) = u(L, t) = 0$

$(1+x)^{-2} XT'' - X''T = 0$

$(1+x)^{-2} \frac{T''}{T} - \frac{X''}{X} = 0$

$\frac{T''}{T} = \frac{X''}{(1+x)^{-2} X} = -\lambda$

$$\boxed{X'' + \frac{\lambda}{(1+x)^2} X = 0} \qquad T'' + \lambda T = 0$$

$X(0) = X(L) = 0$

Try: $\qquad X = (1+x)^a$

$$a(a-1)(1+x)^{a-2} + \frac{\lambda}{(1+x)^2}(x+1)^a = 0$$

$a(a-1) + \lambda = 0$

$$\boxed{\lambda = -a(a-1)}$$

6.2 Boundary Conditions of the Third Kind

Problems

1. Use the method of separation of variables to obtain the ODE's for x and for t for equations (6.2.1) - (6.2.3).

2. Give the details for the case $\lambda > 0$ in solving (6.2.4) - (6.2.6).

3. Discuss

$$\lim_{n\to\infty} \lambda_n$$

for the above problem.

4. Write the Rayleigh quotient for (6.2.4) - (6.2.6) and show that the eigenvalues are all positive. (That means we should have considered only case 3.)

5. What if $h < 0$ in (6.2.3)? Is there an h for which $\lambda = 0$ is an eigenvalue of this problem?

1. $u_t = ku_{xx}$

$u(0, t) = 0 \qquad \Rightarrow \quad X(0) = 0$

$u_x(L, t) = -h\, u\,(L, t) \qquad \Rightarrow X'(L) = -h\, X(L)$

$x\dot{T} = kX''T$

$\frac{\dot{T}}{kT} = \frac{X''}{X} = -\,\lambda$

$\boxed{\dot{T} + k\,\lambda T = 0}$

$$\boxed{\begin{array}{l} X'' + \lambda\, X = 0 \\[1em] X(0) = 0 \\[1em] X'(L) = -h\, X(L) \end{array}}$$

2. $\lambda > 0$

The solution of the ODE for X is

$X = B \cos \sqrt{\lambda} x + A \sin \sqrt{\lambda} x$

$X(0) = 0 \Rightarrow B = 0$

$$X'(x) = A \sqrt{\lambda} \cos \sqrt{\lambda} x + \underbrace{0}_{\substack{\uparrow \\ \text{since } B = 0}}$$

$X'(L) = A \sqrt{\lambda} \cos \sqrt{\lambda} L$

$-h X(L) = -h A \sin \sqrt{\lambda} L$

$\Rightarrow A \sqrt{\lambda} \cos \sqrt{\lambda} L = -h A \sin \sqrt{\lambda} L$

$A \neq 0$ (to avoid trivial solution)

$\Rightarrow \sqrt{\lambda} \cos \sqrt{\lambda} L = -h \sin \sqrt{\lambda} L$

If $\cos \sqrt{\lambda} L = 0 \Rightarrow -h \sin \sqrt{\lambda} L = 0 \Rightarrow \sin \sqrt{\lambda} L = 0$ not possible.
Therefore we can divide by $\cos \sqrt{\lambda} L$

$$\boxed{\tan \sqrt{\lambda} L = -\frac{\sqrt{\lambda}}{h}}$$

3. Graphically we see that the straight line crossing the lower branches of tangent function

since the lower branches are for

$$x \,\epsilon\, \left[\frac{\left(n - \frac{1}{2}\right)\pi}{L}, \frac{n\,\pi}{L} \right] \qquad n = 1,\, 2,\, 3,\, \cdots$$

the eigenvalues are always in these ranges. As n increases, the crossing become closer to the left side (where tan approaches -∞)

$$\Rightarrow \quad \lim_{n \to \infty} \quad \lambda_n = \frac{\left(n - \frac{1}{2}\right)\pi}{L}$$

4. $X'' + \lambda X = 0$

$X(0) = 0$

$X'(L) = -h\, X(L)$

Rayleigh quotient

$$\lambda = \frac{-p\, X\, X'\,|_0^b + \int_0^L \{p[X']^2 - q\, X^2\}\, dx}{\int_0^L \sigma\, X^2\, dx}$$

$$p = 1 \qquad \sigma = 1 \qquad q = 0$$

$$\lambda = \frac{-X\, X'\,|_0^L + \int_0^L (X')^2\, dx}{\int_0^L X^2\, dx}$$

$$-X\, X'\,|_0^L = -X\,(L)\,\underbrace{X'(L)}_{-h\, X(L)} + \underbrace{X(0)}_{=0}\, X'(0) = h\, X(L)^2$$

The numerator is

$$\underbrace{h}_{>0}\,\underbrace{X(L)^2}_{>0} + \underbrace{\int_0^L (X')^2\, dx}_{\geq 0} \quad > 0$$

since $X \neq 0$ since X' maybe zero

The denominator is positive (integrating X^2)

$$\Rightarrow \quad \underline{\lambda > 0}$$

5. If $h < 0$ in (6.2.3)

$$\Rightarrow \quad \text{the numerator is now}$$

$$\underbrace{\underbrace{h}_{<0}\,\underbrace{X(L)^2}_{>0}}_{<0} + \underbrace{\int_0^L (X')^2\,dx}_{\geq 0}$$

$\Rightarrow$ there is a possibility of zero or negative eigenvalues.

For what h one can have zero eigenvalue ?

check (6.2.10)

$B(1 + hL) = 0 \quad \Rightarrow \quad 1 + hL = 0$

$$h = -\frac{1}{L} < 0$$

For this case $B \neq 0 \quad \Rightarrow \quad X_0(x) = Bx$

6.3 Proof of Theorem and Generalizations

Problems

1. Show that if u, v both satisfy the boundary conditions (6.1.9)-(6.1.10) then

$$p\,(uv' - vu')\,|_a^b = 0.$$

2. Show that the right hand side of (6.3.4) is zero even if u, v satisfy periodic boundary conditions, i.e.

$$u(a) = u(b)$$
$$p(a)u'(a) = p(b)u'(b),$$

and similarly for v.

3. What can be proved about eigenvalues and eigenfunctions of the circularly symmetric heat flow problem.
Give details of the proof.
Note: This is a singular Sturm-Liouville problem.

4. Consider the heat flow with convection

$$u_t \;=\; ku_{xx} + V_0\; u_x, \qquad 0 < x < L, \quad t > 0.$$

   a. Show that the spatial ordinary differential equation obtained by separation of variables is <u>not</u> in Sturm-Liouville form.

   b. How can it be reduced to S-L form?

   c. Solve the initial boundary value problem

$$u(0,\,t) = 0, \qquad t > 0,$$

$$u(L,\,t) = 0, \qquad t > 0,$$

$$u(x,\,0) = f(x), \qquad 0 < x < L.$$

1. u, v

$\beta_1\, x(a) \,+\, \beta_2\, x'(a) \,=\, 0$

$\beta_3\, x(b) \,+\, \beta_4\, x'(b) \,=\, 0$

$p(uv' \,-\, vu')\,|_a^b \,=\, 0$

(1). $\quad \beta_2\, v'(a) \,=\, -\,\beta_1\, v(a)$

$\qquad \beta_4\, v'(b) \,=\, -\,\beta_3\, v(b)$

(2). $\quad$ assume $\beta_3 \,\neq\, 0$

$$\left.\begin{array}{l} u(b)\, v'(b) \,=\, -\dfrac{\beta_4}{\beta_3}\, u'(b)\, v'(b) \\[2ex] \underline{-v(b)\, u'(b) \,=\, -\, u'(b)\, \left(-\dfrac{\beta_4}{\beta_3}\, v'(b)\right)} \end{array}\right\} \quad + \,\rightarrow\, 0 \qquad \text{add up to zero}$$

(3). $\quad$ assume $\beta_1 \,\neq\, 0$

$$\left.\begin{array}{l} -u(a)\, v'(a) \,=\, -\left(-\dfrac{\beta_2}{\beta_1}\, u'(a)\right)\, v'(a) \\[2ex] +\, v(a)\, u'(a) \,=\, -\dfrac{\beta_2}{\beta_1}\, v'(a)\, u'(a) \end{array}\right\} \quad + \,\rightarrow\, 0 \qquad \text{add up to zero}$$

(4). $\quad$ If $\beta_3 \,=\, 0 \qquad v'(b) \,=\, 0$

$\qquad\qquad\qquad u'(b) \,=\, 0$

leads to $\qquad u(b)\, v'(b) \,=\, 0$

$\qquad\qquad u'(b)\, v(b) \,=\, 0$

(5). $\quad$ same true if $\beta_1 \,=\, 0$

$\qquad\qquad v'(a) \,=\, 0$

$\qquad\qquad u'(a) \,=\, 0$

2.

$$u(a) \,=\, u(b)$$

$$p(a)\, u'(a) \,=\, p(b)\, u'(b)$$

and similarly for v

$$p(u\, v' \,-\, v\, u')\,\big|_a^b \,=$$

$$= p(b)\, u(b)\, v'(b) \,-\, p(b)\, v(b)\, u'(b)$$

$$\underbrace{-\, p(a)\, \underbrace{u(a)}_{u(b)}\, v'(a)}_{p(b)\, u(b)\, v'(b)} + \underbrace{p(a)\, \underbrace{v(a)}_{v(b)}\, u'(a)}_{p(b)\, u'(b)\, v(b)}$$

this term cancels the one above it.	this matches the term right above it with difference in sign only thus these two terms add up to zero.

4.

$$u_t = k u_{xx} + V_0 u_x$$

a.

$$u = XT$$

$$XT' = kTX'' + V_0 X' T$$

$$\frac{T'}{kT} = \frac{X''}{X} + \frac{V_0}{k}\frac{X'}{X} = -\lambda$$

$$\boxed{X'' + \frac{V_0}{k} X'} + \lambda X = 0$$

The two terms in the box should be combined into one in order to have the equation in Sturm-Liouville form.

b. $X'' + \frac{V_0}{k} X' = (e^{\frac{V_0}{k}x} X)' e^{-\frac{V_0}{k}x}$

(Recall integrating factors!)

Thus the equation becomes

$$(e^{\frac{V_0}{k}x} X')' + \lambda e^{\frac{V_0}{k}x} X = 0$$

This is the Sturm-Liouville form with

$$p = e^{\frac{V_0}{k}x}; \quad q = 0; \quad \sigma = e^{\frac{V_0}{k}x}$$

4c. To solve the initial value problem we have

$$X'' + \frac{V_0}{k} X' + \lambda X = 0$$

$$X(0) = X(L) = 0$$

Try $X = e^{\mu x}$

$$\mu^2 + \frac{V_0}{k} \mu + \lambda = 0$$

$$\mu = \frac{-\frac{V_0}{k} \pm \sqrt{\left(\frac{V_0}{k}\right)^2 - 4\lambda}}{2} = -\frac{V_0}{2k} \pm \sqrt{\left(\frac{V_0}{2k}\right)^2 - \lambda}$$

If $\left(\frac{V_0}{2k}\right)^2 - \lambda > 0$ the solutions are real exponentials which with the boundary conditions yield a trivial solution. Similarly for $\left(\frac{V_0}{2k}\right)^2 = \lambda$, since $X = e^{-\frac{V_0}{2k} x} \, (Ax + B)$ which again is trivial when using the boundary conditions.

If $\left(\frac{V_0}{2k}\right)^2 - \lambda < 0$ then let $\triangle^2 = \lambda - \left(\frac{V_0}{2k}\right)^2$
The solution is $X = e^{-\frac{V_0}{2k} x} (A \cos \triangle x + B \sin \triangle x)$

Using the first boundary condition we get

$$X(0) = 0 = A$$

Thus the second boundary condition gives

$X(L) = 0 = e^{-\frac{V_0}{2k} L} B \sin \triangle L$

$\Rightarrow \triangle L = n\pi \qquad n = 1, 2, \cdots$

$\Rightarrow \lambda_n - \left(\frac{V_0}{2k}\right)^2 = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \cdots$

$$\boxed{\lambda_n = \left(\frac{V_0}{2k}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \cdots}$$

$$\boxed{X_n = e^{-\frac{V_0}{2k}x} \sin \frac{n\pi}{L} x \qquad n = 1, 2, \cdots}$$

Using these eigenvalues in the T equation:

$$T_n' + \lambda_n kT = 0$$

we get

$$T_n = e^{-\lambda_n kt}$$

Thus

$$\boxed{u(x, t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n kt} e^{-\frac{V_0}{2k}x} \sin \frac{n\pi}{L} x \qquad (*)}$$

Use the initial condition:

$$f(x) = \sum_{n=1}^{\infty} b_n e^{-\frac{V_0}{2k}x} \sin \frac{n\pi}{L} x$$

This is a generalized Fourier series of $f(x)$

$$\boxed{b_n = \frac{\int_0^L f(x) e^{-\frac{V_0}{2k}x} \sin \frac{n\pi}{L} x\, dx}{\int_0^L e^{-\frac{V_0}{k}x} \sin^2 \frac{n\pi}{L} x\, dx} \qquad (\#)}$$

The solution is given by (*) with the coefficients by (#), and λ_n in the box above.

6.4 Linearized Shallow Water Equations

Problems

1. Find the second solution of (6.4.13) for $a(c) = -n$.
Hint: Use the power series solution method.

2.
a. Find a relationship between $M(a, b; z)$ and its derivative $\frac{dM}{dz}$.

b. Same for U.

3. Find in the literature a stable recurrence relation to compute the confluent hypergeometric functions.

6.5 Eigenvalues of Perturbed Problems

Problems

1. The flow down a slightly corrugated channel is given by $u(x, y, \epsilon)$ which satisfies

$$\nabla^2 u = -1 \qquad \text{in } |y| \le h(x, \epsilon) = 1 + \epsilon \cos kx$$

subject to

$$u = 0 \qquad \text{on } y = \pm h(x, \epsilon)$$

and periodic boundary conditions in x.

Obtain the first two terms for u.

2. The functions $\phi(x, y, \epsilon)$ and $\lambda(\epsilon)$ satisfy the eigenvalue problem

$$\phi_{xx} + \phi_{yy} + \lambda\phi = 0 \qquad \text{in} \qquad 0 \le x \le \pi, \quad 0 + \epsilon x(\pi - x) \le y \le \pi$$

subject to

$$\phi = 0 \qquad \text{on the boundary.}$$

Find the first order correction to the eigenpair

$$\phi_1^{(0)} = \sin x \sin y$$

$$\lambda_1^{(0)} = 2$$

7 PDEs in Higher Dimensions

7.1 Introduction

7.2 Heat Flow in a Rectangular Domain

Problems

1. Solve the heat equation

$$u_t(x, y, t) = k\left(u_{xx}(x, y, t) + u_{yy}(x, y, t)\right),$$

on the rectangle $0 < x < L, 0 < y < H$ subject to the initial condition

$$u(x, y, 0) = f(x, y),$$

and the boundary conditions

a.

$$u(0, y, t) = u_x(L, y, t) = 0,$$
$$u(x, 0, t) = u(x, H, t) = 0.$$

b.

$$u_x(0, y, t) = u(L, y, t) = 0,$$
$$u_y(x, 0, t) = u_y(x, H, t) = 0.$$

c.

$$u(0, y, t) = u(L, y, t) = 0,$$
$$u(x, 0, t) = u_y(x, H, t) = 0.$$

2. Solve the heat equation on a rectangular box

$$0 < x < L, 0 < y < H, 0 < z < W,$$

$$u_t(x, y, z, t) = k(u_{xx} + u_{yy} + u_{zz}),$$

subject to the boundary conditions

$$u(0, y, z, t) = u(L, y, z, t) = 0,$$

$$u(x, 0, z, t) = u(x, H, z, t) = 0,$$
$$u(x, y, 0, t) = u(x, y, W, t) = 0,$$

and the initial condition

$$u(x, y, z, 0) = f(x, y, z).$$

1. $u_t = k(u_{xx} + u_{yy})$

$u(x, y, 0) = f(x, y)$

$u = X(x)Y(y)T(t)$

$xY\dot{T} = kYX''T + kXTY''$

$$\frac{\dot{T}}{kT} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

$$\dot{T} + \lambda kT = 0 \qquad \frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -\mu$$

$$X'' + \mu X = 0 \qquad Y'' + (\lambda - \mu)Y = 0$$

a. $X(0) = X'(L) = 0$

$Y(0) = Y(H) = 0$

$$\Rightarrow X_n = \sin\frac{\left(n - \frac{1}{2}\right)\pi}{L}x \qquad \mu_n = \left[\frac{\left(n - \frac{1}{2}\right)\pi}{L}\right]^2 \qquad n = 1, 2, \cdots$$

$$\Rightarrow Y_{nm} = \sin\frac{m\pi}{H}y \qquad \lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2 \qquad n = 1, 2, \cdots \qquad m = 1, 2, \cdots$$

$$\lambda_{nm} = \left[\frac{\left(n - \frac{1}{2}\right)\pi}{L}\right]^2 + \left(\frac{m\pi}{H}\right)^2 \qquad n, m = 1, 2, \cdots$$

$T_{nm} = e^{-\lambda_{nm}kt}$

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} a_{nm}\, e^{-\lambda_{nm}kt} \sin\left(n - \frac{1}{2}\right)\frac{\pi}{L}x \sin\frac{m\pi}{H}y$$

$$f(x, y) = u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\left(n - \frac{1}{2}\right)\frac{\pi}{L}x \sin\frac{m\pi}{H}y$$

$$a_{nm} = \frac{\int_0^H \int_0^L f(x, y) \sin\left(n - \frac{1}{2}\right)\frac{\pi}{L}x \sin\frac{m\pi}{H}y\,dx\,dy}{\int_0^H \int_0^L \sin^2\left(n - \frac{1}{2}\right)\frac{\pi}{L}x \sin^2\frac{m\pi}{H}y\,dx\,dy}$$

1b. $X'(0) \, = \, X(L) \, = \, 0$

$Y'(0) \, = \, Y'(H) \, = \, 0$

$$X_n \, = \, \cos \frac{\left(n \, - \, \frac{1}{2}\right) \, \pi}{L} \, x \qquad \mu_n \, = \, \left[\frac{\left(n \, - \, \frac{1}{2}\right) \, \pi}{L} \right]^2 \qquad\qquad n \, = \, 1, \, 2, \, \cdots$$

$$Y_{nm} \, = \, \cos \frac{m \, \pi}{H} \, x \qquad \lambda_{nm} \, = \, \left[\frac{\left(n \, - \, \frac{1}{2}\right) \, \pi}{L} \right]^2 \, + \, \left(\frac{m \, \pi}{H} \right)^2 \qquad n \, = \, 1, \, 2, \, \cdots \qquad m \, =$$
$0, \, 1, \, 2, \, \cdots$

$$u(x,y,t) \, = \, \sum_{n=1}^{\infty} \frac{1}{2} a_{n0} \, e^{-k \, \lambda_{n0} t} \, \cos \frac{\left(n \, - \, \frac{1}{2}\right) \, \pi}{L} \, x + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{nm} \, e^{-k \, \lambda_{nm} t} \, \cos \frac{\left(n \, - \, \frac{1}{2}\right) \, \pi}{L} \, x \, \cos \frac{m \, \pi}{H} \, y$$

$$a_{nm} \, = \, \frac{\int_0^H \int_0^L \, f(x, \, y) \, \cos \left(n \, - \, \frac{1}{2}\right) \, \frac{\pi}{L} \, x \, \cos \, \frac{m \, \pi}{H} \, y \, dx \, dy}{\int_0^H \int_0^L \, \cos^2 \left(n \, - \, \frac{1}{2}\right) \, \frac{\pi}{L} \, x \, \cos^2 \, \frac{m \, \pi}{H} \, y \, dx \, dy} \qquad n \, = \, 1, \, 2, \, \cdots \qquad m \, = \, 0, \, 1, \, 2, \, \cdots$$

1c. $X(0) = X(L) = 0$

$Y(0) = Y'(H) = 0$

$$X_n = \sin \frac{n\pi}{L} x \qquad \mu_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \cdots$$

$$Y_{nm} = \sin \frac{\left(m - \frac{1}{2}\right)\pi}{H} y \qquad \lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left[\frac{\left(m - \frac{1}{2}\right)\pi}{H}\right]^2 \qquad m, n = 1, 2, \cdots$$

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{nn} e^{-k\lambda_{nm} t} \sin \frac{n\pi}{L} x \sin \frac{\left(m - \frac{1}{2}\right)\pi}{H} y$$

$$a_{nm} = \frac{\int_0^L \int_0^H f(x, y) \sin \frac{n\pi}{L} x \sin \frac{\left(m - \frac{1}{2}\right)\pi}{H} y \, dy \, dx}{\int_0^L \int_0^H \sin^2 \frac{n\pi}{L} x \sin^2 \frac{\left(m - \frac{1}{2}\right)\pi}{H} y \, dy \, dx}$$

2. $u_t = k(u_{xx} + u_{yy} + u_{zz})$

$$\dot{T}XYZ = kT(X''YZ + XY''Z + XYZ'')$$

$$\frac{\dot{T}}{kT} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\lambda$$

$$\dot{T} + \lambda kT = 0 \qquad \frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} - \lambda = -\mu$$

$$X'' + \mu X = 0 \qquad \frac{Y''}{Y} = -\frac{Z''}{Z} - \lambda + \mu = -\nu$$
$$X(0) = X(L) = 0$$

$$Y'' + \nu Y = 0$$
$$Y(0) = Y(H) = 0$$

$$Z'' + (\lambda - \mu - \nu)Z = 0$$
$$Z(0) = Z(W) = 0$$

$$X_n = \sin\left(\tfrac{n\pi}{L}\right)x \qquad \mu_n = \left(\tfrac{n\pi}{L}\right)^2 \qquad n = 1, 2, \cdots$$

$$Y_m = \sin\tfrac{m\pi}{H}y \qquad \nu_m = \left(\tfrac{m\pi}{H}\right)^2 \qquad m = 1, 2, \cdots$$

$$Z_{nm\ell} = \sin\tfrac{\ell\pi}{W}z \qquad \lambda_{nm\ell} = \left(\tfrac{n\pi}{L}\right)^2 + \left(\tfrac{m\pi}{H}\right)^2 + \left(\tfrac{\ell\pi}{W}\right)^2 \qquad \ell = 1, 2, \cdots$$

$$u(x,y,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty} a_{nm\ell}\, e^{-k\lambda_{nm\ell}t} \sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y \sin\frac{\ell\pi}{W}z$$

$$a_{nm\ell} = \frac{\int_0^L \int_0^H \int_0^W f(x, y, z) \sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y \sin\frac{\ell\pi}{W}z\,dz\,dy\,dx}{\int_0^L \int_0^H \int_0^W \sin^2\frac{n\pi}{L}x \sin^2\frac{m\pi}{H}y \sin^2\frac{\ell\pi}{W}z\,dz\,dy\,dx}$$

7.3 Vibrations of a rectangular Membrane

Problems

1. Solve the wave equation

$$u_{tt}(x,y,t) = c^2\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right),$$

on the rectangle $0 < x < L, 0 < y < H$ subject to the initial conditions

$$u(x,y,0) = f(x,y),$$

$$u_t(x,y,0) = g(x,y),$$

and the boundary conditions

a.

$$u(0,y,t) = u_x(L,y,t) = 0,$$
$$u(x,0,t) = u(x,H,t) = 0.$$

b.

$$u(0,y,t) = u(L,y,t) = 0,$$
$$u(x,0,t) = u(x,H,t) = 0.$$

c.

$$u_x(0,y,t) = u(L,y,t) = 0,$$
$$u_y(x,0,t) = u_y(x,H,t) = 0.$$

2. Solve the wave equation on a rectangular box

$$0 < x < L, 0 < y < H, 0 < z < W,$$

$$u_{tt}(x,y,z,t) = c^2(u_{xx} + u_{yy} + u_{zz}),$$

subject to the boundary conditions

$$u(0,y,z,t) = u(L,y,z,t) = 0,$$

$$u(x,0,z,t) = u(x,H,z,t) = 0,$$
$$u(x,y,0,t) = u(x,y,W,t) = 0,$$

and the initial conditions

$$u(x,y,z,0) = f(x,y,z),$$
$$u_t(x,y,z,0) = g(x,y,z).$$

3. Solve the wave equation on an isosceles right-angle triangle with side of length a

$$u_{tt}(x, y, t) = c^2(u_{xx} + u_{yy}),$$

subject to the boundary conditions

$$u(x, 0, t) = u(0, y, t) = 0,$$

$$u(x, y, t) = 0, \qquad \text{on the line} \qquad x + y = a$$

and the initial conditions

$$u(x, y, 0) = f(x, y),$$

$$u_t(x, y, 0) = g(x, y).$$

1. $u_{tt} = c^2 (u_{xx} + u_{yy})$

$$\ddot{T}XY = c^2 T (X''Y + XY'')$$

$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

$$\ddot{T} + \lambda c^2 T = 0 \qquad \frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -\mu$$

$$X'' + \mu X = 0 \qquad Y'' + (\lambda - \mu) Y = 0$$

a. $X(0) = X'(L) = 0$

$Y(0) = Y(H) = 0$

as in previous section

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ a_{nm} \cos c\sqrt{\lambda_{nm}}\, t + b_{nm} \sin c\sqrt{\lambda_{nm}}\, t \right\} \sin \frac{\left(n - \frac{1}{2}\right)\pi}{L} x \sin \frac{m\pi}{H} y$$

Initial Conditions

$$f(x, y) = u(x,y,0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{\left(n - \frac{1}{2}\right)\pi}{L} x \sin \frac{m\pi}{H} y \qquad \text{yields } a_{nm}$$

$$a_{nm} = \frac{\int_0^H \int_0^L f(x, y) \sin \left(n - \frac{1}{2}\right) \frac{\pi}{L} x \sin \frac{m\pi}{H} y \, dx\, dy}{\int_0^H \int_0^L \sin^2 \left(n - \frac{1}{2}\right) \frac{\pi}{L} x \sin^2 \frac{m\pi}{H} y \, dx\, dy}$$

$$g(x, y) = u_t(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}\, b_{nm} \sin (n - \frac{1}{2}) \frac{\pi}{L} x \sin \frac{m\pi}{H} y$$

$$b_{nm} = \frac{\int_0^L \int_0^H g(x, y) \sin \frac{\left(n - \frac{1}{2}\right)\pi}{L} x \sin \frac{m\pi}{H} y \, dy\, dx}{c\sqrt{\lambda_{nm}} \int_0^L \int_0^H \sin^2 (n - \frac{1}{2}) \frac{\pi}{L} x \sin^2 \frac{m\pi}{H} y \, dy\, dx}$$

b.

$$X(0) = X(L) = 0$$

$$Y(0) = Y(H) = 0$$

$$X_n = \sin \tfrac{n\pi}{L} x \qquad \mu_n = \left(\tfrac{n\pi}{L}\right)^2 \qquad n = 1, 2, \cdots$$

$$Y_{nm} = \sin \tfrac{m\pi}{H} y \qquad \lambda_{nm} = \left(\tfrac{n\pi}{L}\right)^2 + \left(\tfrac{m\pi}{H}\right)^2 \qquad m = 1, 2, \cdots$$

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ a_{nm} \cos c\sqrt{\lambda_{nm}}\, t + b_{nm} \sin c\sqrt{\lambda_{nm}}\, t \right\} \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y$$

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y$$

$$g(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}\, b_{nm} \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y$$

a_{nm} , b_{nm} in a similar fashion to part a.

$$a_{nm} = \frac{\int_0^H \int_0^L f(x, y) \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y \, dx \, dy}{\int_0^H \int_0^L \sin^2 \frac{n\pi}{L} x \sin^2 \frac{m\pi}{H} y \, dx \, dy}$$

$$b_{nm} = \frac{\int_0^L \int_0^H g(x, y) \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y \, dy \, dx}{c\sqrt{\lambda_{nm}} \int_0^L \int_0^H \sin^2 \frac{n\pi}{L} x \sin^2 \frac{m\pi}{H} y \, dy \, dx}$$

c. see 1b in 7.1

$$X_n = \cos \frac{\left(n - \frac{1}{2}\right)\pi}{L} x \qquad \mu_n = \left[\frac{\left(n - \frac{1}{2}\right)\pi}{L}\right]^2 \qquad n = 1, 2, \cdots$$

$$Y_{nm} = \cos \frac{m\pi}{H} y \qquad \lambda_{nm} = \left[\frac{\left(n - \frac{1}{2}\right)\pi}{L}\right]^2 + \left(\frac{m\pi}{H}\right)^2 \qquad m = 0, 1, 2, \cdots$$

$$u(x,y,t) = \sum_{n=1}^{\infty} \left\{ a_{n0} \cos c\sqrt{\lambda_{n0}}\, t + b_{n0} \sin c\sqrt{\lambda_{n0}}\, t \right\} \cos \frac{\left(n - \frac{1}{2}\right)\pi}{L} x$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ a_{nm} \cos c\sqrt{\lambda_{nm}}\, t + b_{nm} \sin c\sqrt{\lambda_{nm}}\, t \right\} \cos \frac{\left(n - \frac{1}{2}\right)\pi}{L} x \cos \frac{m\pi}{H} y$$

$f(x, y)$ yields $a_{n0},\ a_{nm}$

$g(x, y)$ yields $b_{n0},\ b_{nm}$

2. Since boundary conditions are the same as in 2 section 7.1

$$u(x,y,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty}\left\{a_{nm\ell}\cos c\sqrt{\lambda_{nm\ell}}\,t + b_{nm\ell}\sin c\sqrt{\lambda_{nm\ell}}\,t\right\}\sin\frac{n\pi}{L}x\,\sin\frac{m\pi}{H}y\,\sin\frac{\ell\pi}{W}z$$

$f(x, y, z)$ yields $a_{nm\ell}$

$$a_{nm\ell} = \frac{\int_0^L\int_0^H\int_0^W f(x, y, z)\sin\frac{n\pi}{L}x\,\sin\frac{m\pi}{H}y\,\sin\frac{\ell\pi}{W}z\,dz\,dy\,dx}{\int_0^L\int_0^H\int_0^w \sin^2\frac{n\pi}{L}x\,\sin^2\frac{m\pi}{H}y\,\sin^2\frac{\ell\pi}{W}z\,dz\,dy\,dx}$$

$g(x, y, z)$ yields $b_{nm\ell}$

$$b_{nm\ell} = \frac{\int_0^L\int_0^H\int_0^W g(x, y, z)\sin\frac{n\pi}{L}x\,\sin\frac{m\pi}{H}y\,\sin\frac{\ell\pi}{W}z\,dz\,dy\,dx}{c\sqrt{\lambda_{nm\ell}}\int_0^L\int_0^H\int_0^W \sin^2\frac{n\pi}{L}x\,\sin^2\frac{m\pi}{H}y\,\sin^2\frac{\ell\pi}{W}z\,dz\,dy\,dx}$$

3.
See the solution of Helmholtz equation (problem 2 in section 7.4)

$$\psi_{nm}(x,y) \,=\, \sin\frac{\pi}{a}(m+n)x\,\sin\frac{\pi}{a}ny \,-\, (-1)^m\,\sin\frac{\pi}{a}(m+n)y\,\sin\frac{\pi}{a}nx$$

$$\lambda_{nm} \,=\, \frac{\pi}{a}\sqrt{(m+n)^2 \,+\, n^2} \qquad n, m = 1, 2, \cdots$$

The solution is similar to 1b

$$u(x,y,t) \,=\, \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{ a_{nm}\,\cos c\sqrt{\lambda_{nm}}\,t \,+\, b_{nm}\,\sin\, c\sqrt{\lambda_{nm}}\,t \right\}\psi_{nm}(x,y)$$

$$f(x,\,y) \,=\, \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\,\psi_{nm}(x,y)$$

$$g(x,\,y) \,=\, \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}\,b_{nm}\,\psi_{nm}(x,y)$$

a_{nm} , b_{nm} in a similar fashion to 1a.

$$a_{nm} \,=\, \frac{\int_0^a\int_0^a f(x,\,y)\,\psi_{nm}(x,y)\,dx\,dy}{\int_0^a\int_0^a \psi_{nm}^2(x,y)\,dx\,dy}$$

$$b_{nm} \,=\, \frac{\int_0^a\int_0^a g(x,\,y)\psi_{nm}(x,y)\,dy\,dx}{c\sqrt{\lambda_{nm}}\int_0^a\int_0^a \psi_{nm}^2(x,y)\,dy\,dx}$$

7.4 Helmholtz Equation

Problems

1. Solve

$$\nabla^2\phi + \lambda\phi = 0 \qquad [0,1]\times[0,1/4]$$

subject to

$$\phi(0,y) = 0$$

$$\phi_x(1,y) = 0$$

$$\phi(x,0) = 0$$

$$\phi_y(x,1/4) = 0.$$

Show that the results of the theorem are true.

2. Solve Helmholtz equation on an isosceles right-angle triangle with side of length a

$$u_{xx} + u_{yy} + \lambda u = 0,$$

subject to the boundary conditions

$$u(x,0,t) = u(0,y,t) = 0,$$

$$u(x,y,t) = 0, \qquad \text{on the line} \qquad x + y = a.$$

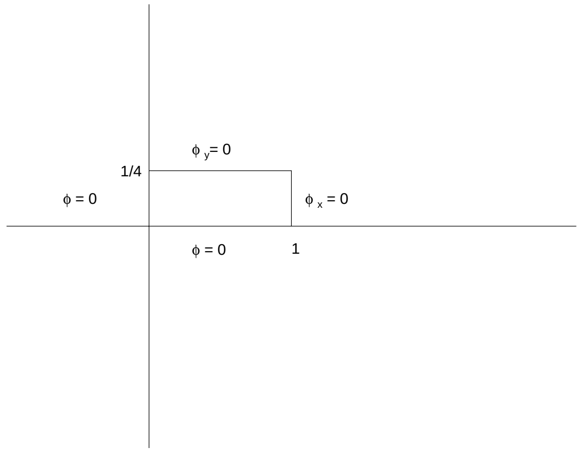


Figure 58: Domain fro problem 1 of 7.4

1.

$\varphi(x, y) = XY$

$X''Y + XY'' + \lambda XY = 0$

$$\frac{X''}{X} = -\lambda - \frac{Y''}{Y} = -\mu$$

$X'' + \mu X = 0 \qquad\qquad Y'' + (\lambda - \mu) Y = 0$

$X(0) = X'(1) = 0 \qquad\qquad Y(0) = Y'(1/4) = 0$

$\Downarrow \qquad\qquad \Downarrow$

$X_n = \sin(n - \frac{1}{2}) \pi x \qquad\qquad Y_{nm} = \sin(m - \frac{1}{2}) 4 \pi y$

$\mu_n = \left[(n - \frac{1}{2}) \pi\right]^2 \qquad\qquad \lambda_{nm} = \left[(n - \frac{1}{2}) \pi\right]^2 + [(4m - 2) \pi]^2$

$n = 1, 2, \cdots \qquad\qquad m = 1, 2, \cdots$

$$\varphi_{nm} = \sin(n - \tfrac{1}{2}) \pi x \ \sin(4m - 2) \pi y$$

$\lambda_{nm} = [(n - \frac{1}{2}) \pi]^2 + [4m - 2) \pi]^2 \qquad n, m = 1, 2, \cdots$

Infinite number of eigenvalues

$\lambda_{11} = \frac{1}{4} \pi^2 + 4\pi^2$ is the smallest.

There is no largest since $\lambda_{nm} \to \infty$ as n, m increase

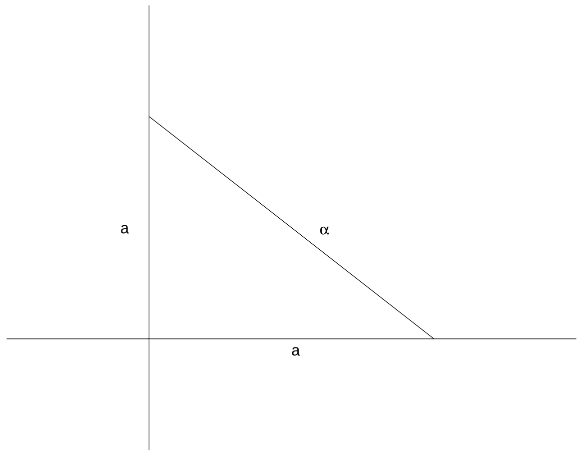


Figure 59: Domain for problem 2 of 7.4

2.

The analysis is more involved when the equation is NOT separable in coordinates suitable for the boundary. Only two nonseparable cases have been solved in detail, one for a boundary which is an isosceles right triangle.

The function

$\sin \dfrac{\mu \pi x}{a} \sin \dfrac{\nu \pi}{a} y$

is zero along the x and y part of the boundary but is not zero along the diagonal side. However, the combination

$\sin \frac{\mu \pi}{a} x \sin \frac{\nu \pi}{a} y \mp \sin \frac{\mu \pi}{a} y \sin \frac{\nu \pi}{a} x$

is zero along the diagonal if μ and ν are integers. (The + sign is taken when $|\mu - \nu|$ is even and the $-$ sign when $|\mu - \nu|$ is odd).

The eigenfunctions

$\psi_{mn}(x, y) = \sin \dfrac{\pi}{a}(m + n)\, x \sin \dfrac{\pi}{a} ny - (-1)^m \sin \dfrac{\pi}{a}(m + n)\, y \sin \dfrac{\pi}{a} nx$

where m, n are positive integers.

The only thing we have to show is the boundary condition on the line $x + y = a$. To show this, rotate by $\pi/4$

$$\Rightarrow x = \frac{1}{\sqrt{2}}(\xi - \eta)$$

$$y = \tfrac{1}{\sqrt{2}}(\xi + \eta)$$

$$\psi_{mn} = \begin{cases} \sin\dfrac{\pi}{\alpha}(m + 2n)\,\xi\,\sin\dfrac{\pi}{\alpha}m\,\eta \quad - \quad \sin\frac{\pi}{\alpha}(m + 2n)\,\eta\,\sin\frac{\pi}{\alpha}m\,\xi & m = 2, 4, \cdots \\ \\ \cos\dfrac{\pi}{\alpha}(m + 2n)\,\eta\,\cos\dfrac{\pi}{\alpha}m\,\xi \quad - \quad \cos\frac{\pi}{\alpha}(m + 2n)\,\xi\,\cos\frac{\pi}{\alpha}m\,\eta & m = 1, 3, \cdots \end{cases}$$

$\Rightarrow \psi_{mn} = 0 \qquad$ for $\xi = \alpha/2 \qquad$ which is $x + y = a$.

The eigenvalues are:

$\lambda_{mn} = \left(\frac{\pi}{a}\right)\sqrt{(m + n)^2 + n^2}$

7.5 Vibrating Circular Membrane

Problems

1. Solve the heat equation

$$u_t(r,\theta,t) = k\nabla^2 u, \qquad 0 \le r < a, 0 < \theta < 2\pi, t > 0$$

subject to the boundary condition

$$u(a,\theta,t) = 0 \qquad \text{(zero temperature on the boundary)}$$

and the initial condition

$$u(r,\theta,0) = \alpha(r,\theta).$$

2. Solve the wave equation

$$u_{tt}(r,t) = c^2(u_{rr} + \frac{1}{r}u_r),$$
$$u_r(a,t) = 0,$$
$$u(r,0) = \alpha(r),$$
$$u_t(r,0) = 0.$$

Show the details.

3. Consult numerical analysis textbook to obtain the smallest eigenvalue of the above problem.

4. Solve the wave equation

$$u_{tt}(r,\theta,t) - c^2\nabla^2 u = 0, \qquad 0 \le r < a, 0 < \theta < 2\pi, t > 0$$

subject to the boundary condition

$$u_r(a,\theta,t) = 0$$

and the initial conditions

$$u(r,\theta,0) = 0,$$
$$u_t(r,\theta,0) = \beta(r)\cos 5\theta.$$

5. Solve the wave equation

$$u_{tt}(r,\theta,t) - c^2\nabla^2 u = 0, \qquad 0 \le r < a, 0 < \theta < \pi/2, t > 0$$

subject to the boundary conditions

$$u(a,\theta,t) = u(r,0,t) = u(r,\pi/2,t) = 0 \qquad \text{(zero displacement on the boundary)}$$

and the initial conditions

$$u(r,\theta,0) = \alpha(r,\theta),$$
$$u_t(r,\theta,0) = 0.$$

1. $u_t = k \nabla^2 u \qquad 0 \le r \le a$

$u(a, \theta, t) = 0 \qquad 0 \le \theta \le 2\pi$

$u(r, \theta, 0) = \alpha(r, \theta) \qquad t > 0$

$$\dot{T} R \Theta = kT \left[\Theta \tfrac{1}{r}(r R')' + \tfrac{1}{r^2} R \Theta''\right]$$

$$\frac{\dot{T}}{kT} = \frac{1}{R\,r}(r R')' + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda$$

$$\dot{T} + k\lambda T = 0 \qquad \frac{r}{R}(r R')' + \frac{\Theta''}{\Theta} = -\lambda r^2$$

$$\frac{r}{R}(r R')' + \lambda r^2 = -\frac{\Theta''}{\Theta} = \mu$$

$$\Theta'' + \mu \Theta = 0 \qquad r(r R')' + (\lambda r^2 - \mu) R = 0$$

$$\Theta(0) = \Theta(2\pi) \qquad |R(0)| < \infty$$

$$\Theta'(0) = \Theta'(2\pi) \qquad R(a) = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\mu_m = m^2 \qquad \boxed{R_m = C_{1m} J_m(\sqrt{\lambda}\,r) \text{ to satisfy } |R(0)| < \infty}$$

$$\Theta_m = \begin{cases} \sin m\theta \\ \cos m\theta \end{cases} \quad m = 1, 2, \cdots$$

$$R_m(\sqrt{\lambda}\,a) = C_m J_m(\sqrt{\lambda}\,a) = 0$$

$$\mu_0 = 0 \qquad \Downarrow$$

$\Theta_0 = 1 \qquad \lambda_{nm}$ are solutions of $\boxed{J_m(\sqrt{\lambda}\,a) = 0}$

$$u(r,\,\theta,\,t) \;=\; \sum_{m=1}^{\infty}\;\sum_{n=1}^{\infty}\left(a_{nm}\,\cos\,m\,\theta\,+\,b_{nm}\,\sin\,m\,\theta\right)e^{-k\,\lambda_{nm}t}\,J_m\,(\sqrt{\lambda_{nm}}\,r)$$

$$+\;\underbrace{\sum_{n=1}^{\infty}\,a_{n0}\,\cdot\,\underbrace{1}_{=\Theta_0}\;e^{-k\,\lambda_{n0}t}\,J_0\,(\sqrt{\lambda_{n0}}\,r)}_{m=0}$$

$$\alpha\,(r,\,\theta)\;=\;\sum_{n=1}^{\infty}\,a_{n0}\,J_0\,(\sqrt{\lambda_{n0}}\,r)\;+\;\sum_{m=1}^{\infty}\;\sum_{n=1}^{\infty}\left(a_{nm}\,\cos\,m\,\theta\,+\,b_{nm}\,\sin\,m\,\theta\right)J_m\,(\sqrt{\lambda_{nm}}\,r)$$

$$a_{n0}\;=\;\frac{\int_0^{2\pi}\;\int_0^a\;\alpha\,(r,\,\theta)\,J_0\,(\sqrt{\lambda_{n0}}\,r)\,rd\,rd\theta}{\int_0^{2\pi}\;\int_0^a\;J_0^2\,(\sqrt{\lambda_{n0}}\,r)\,rd\,rd\theta}$$

$$a_{nm}\;=\;\frac{\int_0^{2\pi}\;\int_0^a\;\alpha\,(r,\,\theta)\,\cos\;m\,\theta\,J_m\,(\sqrt{\lambda_{nm}}\,r)\,rd\,rd\theta}{\int_0^{2\pi}\;\int_0^a\;\cos^2\;m\,\theta\,J_m^2\,(\sqrt{\lambda_{nm}}\,r)\,rd\,rd\theta}$$

$$b_{nm}\;=\;\frac{\int_0^{2\pi}\;\int_0^a\;\alpha\,(r,\,\theta)\,\sin\;m\,\theta\,J_m\,(\sqrt{\lambda_{nm}}\,r)\,rd\,rd\theta}{\int_0^{2\pi}\;\int_0^a\;\sin^2\;m\,\theta\,J_m^2\,(\sqrt{\lambda_{nm}}\,r)\,rd\,rd\theta}$$

2. $u_{tt} - c^2 \left(u_{rr} + \frac{1}{r} u_r\right)$

$$u_r(a, t) = 0$$

$$u(r, 0) = \alpha(r)$$

$$u_t(r, 0) = 0$$

$$\ddot{T} R - c^2 \left(R'' + \frac{1}{r} R'\right) T = 0$$

$$\frac{\ddot{T}}{c^2 T} = \frac{R'' + \frac{1}{r} R'}{R} = -\lambda$$

$$\ddot{T} + \lambda c^2 T = 0 \qquad\qquad \underbrace{R'' + \frac{1}{r} R'}_{} + \lambda R = 0$$

$$\frac{1}{r} (r R')' + \lambda R = 0$$

multiply by r^2

$$\left.\begin{aligned} & r(r R')' + \lambda r^2 R = 0 \\ & |R(0)| < \infty \\ & R'(a) = 0 \end{aligned}\right\}$$

This is Bessel's equation with $\mu = 0$

$\Rightarrow R_n(r) = J_0(\sqrt{\lambda_n}\, r)$

where $\sqrt{\lambda_n}\, J_0'(\sqrt{\lambda_n}\, a) = 0$

gives the eigenvalues λ_n

$$u(r,t) = \sum_{n=1}^{\infty} \left\{ a_n \cos \sqrt{\lambda_n}\, ct + b_n \sin c \sqrt{\lambda_n}\, t \right\} J_0(\sqrt{\lambda_n}\, r)$$

$$\alpha(r) = \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n}\, r)$$

This yields a_n. $\Rightarrow a_n = \dfrac{\int_0^a \alpha(r) J_0(\sqrt{\lambda_n}\, r)\, r dr}{\int_0^a J_0^2(\sqrt{\lambda_n}\, r)\, r dr}$

$$0 = u_t(r, 0) = \sum_{n=1}^{\infty} c \sqrt{\lambda_n}\, b_n J_0(\sqrt{\lambda_n}\, r) \Rightarrow \underline{b_n = 0}$$

4. $u_{tt} - c^2 \nabla^2 u = 0$

$u_r(a, \theta, t) = 0$

$u(r, \theta, 0) = 0$

$u_t(r, \theta, 0) = \beta(r) \cos 5\theta$

$$\Downarrow$$

$\ddot{T} + \lambda c^2 T = 0$ $\qquad$ $\Theta'' + \mu \Theta = 0$ $\qquad$ $r(r R')' + (\lambda r^2 - \mu) R = 0$

$T(0) = 0$ $\qquad$ $\Theta(0) = \Theta(2\pi)$ $\qquad$ $|R(0)| < \infty$

$\Theta'(0) = \Theta'(2\pi)$ $\qquad$ $R'(a) = 0$

$T = a \cos c\sqrt{\lambda_{nm}}\, t$ $\qquad$ $\Downarrow$

$+ b \sin c\sqrt{\lambda_{nm}}\, t$ $\qquad$ $\mu_0 = 0 \quad \Theta_0 = 1$ $\qquad$ $R = J_n(\sqrt{\lambda}\, r)$

Since $T(0) = 0$ $\qquad$ $R'(a) = J_n'(\sqrt{\lambda}\, a) \cdot \sqrt{\lambda} = 0$

$\mu_n = n^2 \quad \Theta_m = \begin{cases} \cos n\theta \\ \sin n\theta \end{cases}$

$\Downarrow$ $\qquad$ $\Downarrow$

$T = \sin c\sqrt{\lambda_{nm}}\, t$ $\qquad$ $\lambda_{n0} = 0$

or

$J_n'(\sqrt{\lambda_{nm}}\, a) = 0$

$m = 1, 2, \cdots$

for each $n = 0, 1, 2, \cdots$

$$u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \{a_{nm} \cos n\theta + b_{nm} \sin n\theta\} \left\{ J_n(\sqrt{\lambda_{nm}}\, r) \right\} \sin c\sqrt{\lambda_{nm}}\, t$$

$$u_t(r, \theta, 0) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \{a_{nm} \cos n\theta + b_{nm} \sin n\theta\} \; J_n(\sqrt{\lambda_{nm}\, r})\, c\sqrt{\lambda_{nm}} \underbrace{\cos c\sqrt{\lambda_{nm}}\, t}_{=1 \quad \text{at } t=0}$$

Since $u_t(r, \theta, 0) = \beta(r) \cos 5\theta$ all $\sin n\theta$ term should vanish i.e. $b_n = 0$ and all $a_n = 0$ except a_5 $(n = 5)$

$$\beta(r) \cos 5\theta = \sum_{m=0}^{\infty} a_{5m} \cos 5\theta\, J_5(\sqrt{\lambda_{5m}\, r})\, c\sqrt{\lambda_{5m}}$$

This is a generalized Fourier series for $\beta(r)$

$$a_{5m}\, c\sqrt{\lambda_{5m}} \;=\; \frac{\int_0^a \;\beta\,(r)\, J_5\,(\sqrt{\lambda_{5m}\, r)\, rdr}}{\int_0^a \; J_5^2\,(\sqrt{\lambda_{5m}\, r)\, rdr}}$$

$$u(r,\, \theta,\, t) \;=\; \sum_{m=0}^{\infty} a_{5m}\, \cos\, 5\,\theta\, J_5\,(\sqrt{\lambda_{5m}\, r})\, \sin\, c\sqrt{\lambda_{5m}}\, t$$

where $\quad \lambda_{5m}$ can be found from

$$\sqrt{\lambda_{5m}}\, J_5'\,(\sqrt{\lambda_{5m}\, a}) \;=\; 0$$

and a_{5m} from

$$a_{5m} \;=\; \frac{\int_0^a \;\beta\,(r)\, J_5\,(\sqrt{\lambda_{5m}\, r)\, rdr}}{c\sqrt{\lambda_{5m}}\, \int_0^a \; J_5^2\,(\sqrt{\lambda_{5m}\, r)\, rdr}}$$

5. $u_{tt} - c^2 \nabla^2 u = 0$

$u(a, \theta, t) = 0$

$u(r, 0, t) = u(r, \pi/2, t) = 0$

$\ddot{T} + \lambda c^2 T = 0 \qquad \Theta'' + \mu \Theta = 0 \qquad r(r R')' + (\lambda r^2 - \mu) R = 0$

$\Theta(0) = \Theta(\pi/2) = 0 \qquad |R(0)| < \infty$

$R(a) = 0$

$\Downarrow$

$\mu_n = (2n)^2 \qquad \Downarrow$

$\Theta_n = \sin 2n\theta \qquad R(r) = J_{2n}(\sqrt{\lambda_{2n\,m}}\, r)$

$n = 1, 2, \cdots \qquad \boxed{J_{2n}(\sqrt{\lambda_{2n,\,m}}\, a) = 0 \;\; m = 1, 2, \cdots}$

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} J_{2n}(\sqrt{\lambda_{2n,\,m}}\, r) \sin 2n\theta \underbrace{\cos c \sqrt{\lambda_{2n,\,m}}\, t}_{since\, u_t(r,\theta,0)=0}$$

$$g(r, \theta) = u(r, \theta, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} J_{2n}(\sqrt{\lambda_{2n,\,m}}\, r) \sin 2n\theta$$

$$a_{mn} = \frac{\int_0^a \int_0^{\pi/2} J_{2n}(\sqrt{\lambda_{2n,\,m}}\, r)\, g(r, \theta) \sin 2n\theta\, r\, d\theta\, dr}{\int_0^{\pi/2} \int_0^a J_{2n}^2(\sqrt{\lambda_{2n,\,m}}\, r) \sin^2 2n\theta\, r\, d r\, d\theta}$$

7.6 Laplace's Equation in a Circular Cylinder

Problems

1. Solve Laplace's equation

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0, \qquad 0 \le r < a, 0 < \theta < 2\pi, 0 < z < H$$

subject to each of the boundary conditions

a.
$$u(r, \theta, 0) = \alpha(r, \theta)$$
$$u(r, \theta, H) = u(a, \theta, z) = 0$$

b.
$$u(r, \theta, 0) = u(r, \theta, H) = 0$$
$$u_r(a, \theta, z) = \gamma(\theta, z)$$

c.
$$u_z(r, \theta, 0) = \alpha(r, \theta)$$
$$u(r, \theta, H) = u(a, \theta, z) = 0$$

d.
$$u(r, \theta, 0) = u_z(r, \theta, H) = 0$$
$$u_r(a, \theta, z) = \gamma(z)$$

2. Solve Laplace's equation

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0, \qquad 0 \le r < a, 0 < \theta < \pi, 0 < z < H$$

subject to the boundary conditions

$$u(r, \theta, 0) = 0,$$
$$u_z(r, \theta, H) = 0,$$
$$u(r, 0, z) = u(r, \pi, z) = 0,$$
$$u(a, \theta, z) = \beta(\theta, z).$$

3. Find the solution to the following steady state heat conduction problem in a box

$$\nabla^2 u = 0, \qquad 0 \le x < L, 0 < y < L, 0 < z < W,$$

subject to the boundary conditions

$$\frac{\partial u}{\partial x} = 0, \qquad x = 0, x = L,$$
$$\frac{\partial u}{\partial y} = 0, \qquad y = 0, y = L,$$
$$u(x, y, W) = 0,$$
$$u(x, y, 0) = 4\cos\frac{3\pi}{L}x\cos\frac{4\pi}{L}y.$$

4. Find the solution to the following steady state heat conduction problem in a box

$$\nabla^2 u = 0, \qquad\qquad 0 \le x < L, 0 < y < L, 0 < z < W,$$

subject to the boundary conditions

$$\frac{\partial u}{\partial x} = 0, \qquad x = 0, x = L,$$
$$\frac{\partial u}{\partial y} = 0, \qquad y = 0, y = L,$$
$$u_z(x, y, W) = 0,$$
$$u_z(x, y, 0) = 4\cos\frac{3\pi}{L}x\cos\frac{4\pi}{L}y.$$

5. Solve the heat equation inside a cylinder

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}, \qquad\qquad 0 \le r < a,\ 0 < \theta < 2\pi,\ 0 < z < H$$

subject to the boundary conditions

$$u(r, \theta, 0) = u(r, \theta, H) = 0,$$
$$u(a, \theta, z, t) = 0,$$

and the initial condition

$$u(r, \theta, z, 0) = f(r, \theta, z).$$

1. $\dfrac{1}{r}(ru_r)_r + \dfrac{1}{r^2}u_{\theta\theta} + u_{zz} = 0$

(a)

$\Theta'' + \mu\Theta = 0$ $\qquad$ $Z'' - \lambda Z = 0$ $\qquad$ $r(rR')' + (\lambda r^2 - \mu)R = 0$

$\Theta(0) = \Theta(2\pi)$ $\qquad$ $Z(H) = 0$ $\qquad$ $|R(0)| < \infty$

$\Theta'(0) = \Theta'(2\pi)$ $\qquad$ $R(a) = 0$

$\Downarrow$ $\qquad$ $\Downarrow$

$\mu_0 = 0$ $\qquad$ $R_{nm} = J_m(\sqrt{\lambda_{nm}}\, r)$

$\Theta_0 = 1$ $\qquad$ satisfies boundedness

$\mu_m = m^2$

$$\Theta_m = \begin{cases} \sin m\theta \\ \cos m\theta \end{cases}$$

$J_m(\sqrt{\lambda_{nm}}\, a) = 0$

yields eigenvalues

$m = 1, 2, \cdots$ $\qquad$ $\Downarrow$ $\qquad$ $n = 1, 2, \cdots$

$\underline{\lambda > 0!}$

$$Z_{nm} = sinh\sqrt{\lambda_{nm}}\,(z - H)$$

vanishes at $z = H$

$$u(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\cos m\theta + b_{nm}\sin m\theta)\sinh\sqrt{\lambda_{nm}}\,(z - H)\,J_m(\sqrt{\lambda_{nm}}\, r)$$

$\uparrow$

This is zero for $m = 0$

$$\alpha(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\cos m\theta + b_{nm}\sin m\theta)\underbrace{\sinh\sqrt{\lambda_{nm}}\,(-H)}_{\text{this is a constant}}J_m(\sqrt{\lambda_{nm}}\, r)$$

$$a_{nm} = \frac{\int_0^a \int_0^{2\pi} \alpha(r,\theta)\cos m\theta\, J_m(\sqrt{\lambda_{nm}}\, r)\, r d\theta\, dr}{\sinh\sqrt{\lambda_{nm}}\,(-H)\int_0^a \int_0^{2\pi} \cos^2 m\theta\, J_m^2(\sqrt{\lambda_{nm}}\, r)\, r d\theta\, dr}$$

$$b_{nm} = \frac{\int_0^a \int_0^{2\pi} \alpha(r,\theta)\sin m\theta\, J_m(\sqrt{\lambda_{nm}}\, r)\, r d\theta\, dr}{\sinh\sqrt{\lambda_{nm}}\,(-H)\int_0^a \int_0^{2\pi} \sin^2 m\theta\, J_m^2(\sqrt{\lambda_{nm}}\, r)\, r d\theta\, dr}$$

1 b.

$$\Theta'' + \mu\,\Theta = 0 \qquad Z'' - \lambda\,Z = 0 \qquad r(r\,R')' + (\lambda\,r^2 - \mu)\,R = 0$$

$$Z(0) = Z(H) = 0 \qquad |\,R(0)\,| < \infty$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\mu_m = m^2 \qquad Z_n = \sin\tfrac{n\,\pi}{H}\,z$$

$$\Theta_m = \begin{cases} \sin\,m\,\theta \\ \\ \cos\,m\,\theta \end{cases} \qquad \lambda_n = -\left(\tfrac{n\,\pi}{H}\right)^2$$

$$m = 1, 2, \cdots \qquad r(r\,R')' + \left(-\left(\frac{n\,\pi}{H}\right)^2 r^2 - m^2\right) R$$

$$\mu_0 = 0$$

$$\uparrow$$

extra minus sign

$$\Theta_0 = 1$$

$$\Downarrow$$

$$R_{nm} = I_m\left(\tfrac{n\,\pi}{H}\,r\right)$$

$$u(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\cos\,m\,\theta + b_{nm}\sin\,m\,\theta)\sin\frac{n\,\pi}{H}\,z\;I_m\left(\frac{n\,\pi}{H}\,r\right)$$

$$u_r\,(a,\,\theta,\,z) = \gamma\,(\theta,\,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\cos\,m\,\theta + b_{nm}\sin\,m\,\theta)\sin\frac{n\,\pi}{H}\,z\cdot\underbrace{\frac{n\,\pi}{H}\,I'_m\left(\frac{n\,\pi}{H}\,a\right)}_{\text{constant}}$$

$$a_{nm}\,\frac{n\,\pi}{H}\,I'_m\left(\frac{n\,\pi}{H}\,a\right) = \frac{\int_0^{2\pi}\int_0^H \gamma\,(\theta,\,z)\cos\,m\,\theta\,\sin\frac{n\pi}{H}\,z\,dz\,d\theta}{\int_0^{2\pi}\int_0^H \cos^2\,m\,\theta\,\sin^2\,\frac{n\pi}{H}\,z\,dz\,d\theta}$$

$$a_{nm} = \frac{\int_0^{2\pi}\int_0^H \gamma\,(\theta,\,z)\cos\,m\,\theta\,\sin\frac{n\pi}{H}\,z\,dz\,d\theta}{\frac{n\,\pi}{H}\,I'_m\left(\frac{n\,\pi}{H}\,a\right)\int_0^{2\pi}\int_0^H \cos^2\,m\,\theta\,\sin^2\,\frac{n\,\pi}{H}\,z\,dz\,d\theta}$$

$$b_{nm} = \frac{\int_0^{2\pi}\int_0^H \gamma\,(\theta,\,z)\sin\,m\,\theta\,\sin\frac{n\,\pi}{H}\,z\,dz\,d\theta}{\frac{n\,\pi}{H}\,I'_m\left(\frac{n\,\pi}{H}\,a\right)\int_0^{2\pi}\int_0^H \sin^2\,m\,\theta\,\sin^2\,\frac{n\,\pi}{H}\,z\,dz\,d\theta}$$

1 c.

$$\Theta'' + \mu\,\Theta = 0 \qquad\qquad Z'' - \lambda\,Z = 0 \qquad\qquad r(r\,R')' + (\lambda\,r^2 - \mu)\,R = 0$$

$$Z(H) = 0 \qquad\qquad |\,R\,(0)\,| < \infty$$

$$R(a) = 0$$

Solution as in 1a exactly !

But

$$u_z(r,\theta,0) = \alpha\,(r,\,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\,\cos\,m\,\theta + b_{nm}\,\sin\,m\,\theta)\,J_m\,(\sqrt{\lambda_{nm}}\,r)\,\sqrt{\lambda_{nm}}\,\cosh\,\sqrt{\lambda_{nm}}\,(-H)$$

$$a_{nm}\,\sqrt{\lambda_{nm}}\,\cosh\,\sqrt{\lambda_{nm}}\,(-H) = \frac{\int_0^{2\pi}\int_0^a \alpha\,(r,\,\theta)\,\cos\,m\,\theta\,J_m\,(\sqrt{\lambda_{nm}}\,r)\,r d\,r d\theta}{\int_0^{2\pi}\int_0^a \cos^2\,m\,\theta\,J_m^2\,(\sqrt{\lambda_{nm}}\,r)\,r d\,r d\theta}$$

$$a_{nm} = \frac{\int_0^{2\pi}\int_0^a \alpha\,(r,\,\theta)\,\cos\,m\,\theta\,J_m\,(\sqrt{\lambda_{nm}}\,r)\,r d\,r d\theta}{\sqrt{\lambda_{nm}}\,\cosh\,\sqrt{\lambda_{nm}}\,(-H)\,\int_0^{2\pi}\int_0^a \cos^2\,m\,\theta\,J_m^2\,(\sqrt{\lambda_{nm}}\,r)\,r d\,r d\theta}$$

$$b_{nm} = \frac{\int_0^{2\pi}\int_0^a \alpha\,(r,\,\theta)\,\sin\,m\,\theta\,J_m\,(\sqrt{\lambda_{nm}}\,r)\,r d\,r d\theta}{\sqrt{\lambda_{nm}}\,\cosh\,\sqrt{\lambda_{nm}}\,(-H)\,\int_0^{2\pi}\int_0^a \sin^2\,m\,\theta\,J_m^2\,(\sqrt{\lambda_{nm}}\,r)\,r d\,r d\theta}$$

1 d.

$\Theta'' + \mu\Theta = 0$ $\qquad$ $Z'' - \lambda Z = 0$ $\qquad$ $r(r\,R')' + (\lambda r^2 - \mu)\,R = 0$

$\Downarrow$ $\qquad$ $Z(0) = Z'(H) = 0$ $\qquad$ $|\,R\,(0)\,| < \infty$

$\Downarrow$ $\qquad$ $\Downarrow$

as before $\qquad$ $\lambda_n = -\left[\left(n - \frac{1}{2}\right)\frac{\pi}{H}\right]^2$

$Z_n = \sin\left(n - \frac{1}{2}\right)\frac{\pi}{H}\,z \qquad R_{nm} = I_m\left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}\,r\right)$

$$u\,(r,\,\theta,\,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\cos m\,\theta + b_{nm}\sin m\,\theta)\sin\left(n - \frac{1}{2}\right)\frac{\pi}{H}z\,I_m\left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}\,r\right)$$

$$u_r(a,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\cos m\,\theta + b_{nm}\sin m\,\theta)\sin\left(n - \frac{1}{2}\right)\frac{\pi}{H}\,z\cdot\left(n - \frac{1}{2}\right)\frac{\pi}{H}\,I'_m\left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}\,a\right)$$

Since $u_r(a,\theta,z) = \gamma(z)$ is independent of θ, we must have no terms with θ in the above expansion, that is $b_{nm} = 0$ for all n, m and $a_{nm} = 0$ for all $n, m \geq 1$. Thus $a_{1\,0} \neq 0$

$$\gamma(z) = a_{1\,0}\sin\frac{\pi}{2H}z\,\frac{\pi}{2H}\,I'_0\left(\frac{\pi}{2H}a\right)$$

$$a_{1\,0} = \frac{\int_0^H \gamma\,(z)\,\sin\,\frac{\pi}{2H}\,z\,dz}{\frac{\pi}{2H}\,I'_0\left(\frac{\pi}{2H}a\right)\int_0^H \sin^2\,\frac{\pi}{2H}\,z\,dz}$$

And the solution is

$$u(r,\theta,z) = a_{1\,0}\sin\frac{\pi}{2H}z\,I_0\left(\frac{\pi}{2H}r\right)$$

2. $u(r,\theta,0) = 0 \quad = u_z(r,\theta,H)$

$u(r,0,z) = 0 \quad = u(r,\pi,z)$

$u(a,\theta,z) \quad = \beta(\theta,z)$

$u = R(r)\,\Theta(\theta)\,Z(z)$

$Z'' + \lambda Z = 0 \qquad \Theta'' + \mu\,\Theta = 0 \qquad r(r\,R')' + (\lambda r^2 - \mu)\,R = 0$

$Z(0) = 0 \qquad \Theta(0) = \Theta(\pi) = 0 \qquad |R(0)| < \infty$

$Z'(H) = 0 \qquad \Downarrow$

$\Downarrow$

$Z_n = \sin\left(n - \frac{1}{2}\right)\frac{\pi}{H}z \qquad \Theta_m = \sin m\theta \qquad r(r\,R')' - \left\{\left[\left(n - \frac{1}{2}\right)\frac{\pi}{H}\right]^2 r^2 + m^2\right\} R$

$\mu_m = m^2$

$\lambda_n = \left[\left(n - \dfrac{1}{2}\right)\dfrac{\pi}{H}\right]^2 \qquad m = 1, 2, \cdots \qquad R(r) = I_m\left(\left(n - \dfrac{1}{2}\right)\dfrac{\pi}{H}r\right)$

$n = 1, 2, \cdots$

$$u(r,\theta,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\,I_m\left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}r\right)\sin m\theta\,\sin\left(n - \frac{1}{2}\right)\frac{\pi}{H}z$$

at $r = a$

$$\beta(\theta,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \underbrace{c_{mn}\,I_m\left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}a\right)}_{\text{coefficient of expansion}}\sin m\theta\,\sin\left(n - \frac{1}{2}\right)\frac{\pi}{H}z$$

$$c_{mn} = \frac{\int_0^{\pi}\int_0^{H}\beta(\theta,z)\,\sin m\theta\,\sin\left(n - \frac{1}{2}\right)\frac{\pi}{H}z\,dz\,d\theta}{I_m\left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}a\right)\int_0^{\pi}\int_0^{H}\sin^2 m\theta\,\sin^2\left(n - \frac{1}{2}\right)\frac{\pi}{H}z\,dz\,d\theta}$$

3. $u_{xx} + u_{yy} + u_{zz} = 0$

$$BC: \quad u_x(0, y, z) = 0$$
$$u_x(L, y, z) = 0$$
$$u_y(x, 0, z) = 0$$
$$u_y(x, L, z) = 0$$
$$u(x, y, 0) = 4\cos\frac{3\pi}{L}x\cos\frac{4\pi}{L}y$$

$$u(x, y, z) = X(x)Y(y)Z(z)$$
$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ BC: X'(0) = X'(L) = 0 \end{cases} \Rightarrow \boxed{\begin{array}{l} \lambda_n = \left(\frac{n\pi}{L}\right)^2 \\ X_n = \cos\frac{n\pi}{L}x \qquad n = 0, 1, 2\cdots \end{array}}$$

$$\frac{Y''}{Y} = \lambda - \frac{Z''}{Z} = -\mu$$

$$\begin{cases} Y'' + \mu Y = 0 \\ BC: Y'(0) = Y'(L) = 0 \end{cases} \Rightarrow \boxed{\begin{array}{l} \mu_n = \left(\frac{m\pi}{L}\right)^2 \\ Y_m = \cos\frac{m\pi}{L}y \quad m = 0, 1, 2\cdots \end{array}}$$

$$\frac{Z''}{Z} = \lambda + \mu$$

$$\begin{cases} Z'' - (\lambda + \mu)Z = 0 \\ BC: Z(W) = 0 \end{cases} \qquad Z'' - \left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2\right] Z = 0$$

$$Z_{nm} = \sinh\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2}\,(z - W)$$

general solution

$$u(x,\, y,\, z) \;=\; \sum_{m=0}^{\infty}\; \sum_{n=0}^{\infty} A_{mn} \cos \frac{n\pi}{L}\, x \,\cos \frac{m\pi}{L}\, y \,\sinh \sqrt{\left(\frac{n\pi}{L}\right)^2 \,+\, \left(\frac{m\pi}{L}\right)^2}\, (z \,-\, W)$$

$$u(x,\, y,\, 0) \;=\; \sum_{m=0}^{\infty}\; \sum_{n=0}^{\infty} - A_{mn} \cos \frac{n\pi}{L}\, x \,\cos \frac{m\pi}{L}\, y \,\sinh \sqrt{\left(\frac{n\pi}{L}\right)^2 \,+\, \left(\frac{m\pi}{L}\right)^2}\, W$$

But $u(x, y, 0) \;=\; 4 \cos \frac{3\pi}{L}\, x \,\cos \frac{4\pi}{L}\, y$

Comparing coefficients

$A_{mn} \;=\; 0 \qquad$ for $m \,\neq\, 4 \quad$ or $n \,\neq\, 3$

For $n \;=\; 3;\; m \;=\; 4 \qquad - \, A_{43} \sinh \sqrt{\dfrac{9\pi^2}{L^2} \,+\, \dfrac{16\pi^2}{L^2}}\, W \;=\; 4$

$$- \, A_{43} \sinh \frac{5\pi}{L}\, W \;=\; 4$$

$$A_{43} \;=\; -\,\frac{4}{\sinh \frac{5\pi}{L}\, W}$$

$$u(x,\, y,\, z) \;=\; -\,\frac{4}{\sinh \frac{5\pi}{L}\, W} \cos \frac{3\pi}{L}\, x \,\cos \frac{4\pi}{L}\, y \,\sinh \frac{5\pi}{L}\, (z \,-\, W)$$

4. $u_{xx} + u_{yy} + u_{zz} = 0$

$$u_x(0, y, z) = 0 \quad , \quad u_x(L, y, z) = 0$$

$$u_y(x, 0, z) = 0 \quad , \quad u_y(x, L, z) = 0$$

$$u(x, y, W) = 0$$

$$u(x, y, 0) = 4 \cos \frac{3\pi}{L} x \cos \frac{4\pi}{L} y$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ BC: X'(0) = X'(L) = 0 \end{cases} \Rightarrow \boxed{\begin{array}{l} \lambda_n = \left(\frac{n\pi}{L}\right)^2 \\ X_n = \cos \frac{n\pi}{L} x \end{array}} \quad n = 0, 1, 2 \cdots$$

$$\begin{cases} Y'' + \mu Y = 0 \\ BC: Y'(0) = Y'(L) = 0 \end{cases} \Rightarrow \boxed{\begin{array}{l} \mu_m = \left(\frac{m\pi}{L}\right)^2 \\ Y_m = \cos \frac{m\pi}{L} y \end{array}} \quad m = 0, 1, 2 \cdots$$

$$Z'' - (\lambda + \mu) Z = 0$$

$$Z'(W) = 0$$

$$Z'' - \left[(\frac{n\pi}{L})^2 + (\frac{m\pi}{L})^2 \right] Z = 0$$

$$\boxed{Z_{nm} = \cosh \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2} (z - W)}$$

$$u(x,\, y,\, z) \;=\; A_{00} \,+\, \sum_{m=1}^{\infty} A_{m0} \cos \frac{m\pi}{L}\, y \cosh \frac{m\pi}{L}\,(z \,-\, W)$$

$$+ \sum_{n=1}^{\infty} A_{0n} \cos \frac{n\pi}{L}\, x \cos \cosh \frac{n\pi}{L}\,(z \,-\, W)$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos \frac{n\pi}{L}\, x \cos \frac{m\pi}{L}\, y \cosh \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2}\,(z \,-\, W)$$

$$u_z(x,y,z) \;=\; \sum_{m=1}^{\infty} \frac{m\pi}{L} A_{m0} \cos \frac{m\pi}{L}\, y \sinh \frac{m\pi}{L}\,(z \,-\, W)$$

$$+ \sum_{n=1}^{\infty} \frac{n\pi}{L} A_{0n} \cos \frac{n\pi}{L}\, x \sinh \frac{n\pi}{L}\,(z \,-\, W)$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2}\, A_{mn} \cos \frac{n\pi}{L}\, x \cos \frac{m\pi}{L}\, y \sinh \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2}\,(z - W)$$

At $z \,=\, 0 \qquad u_z(x,y,0) \;=\; -\sum_{m=1}^{\infty} A_{m0} \frac{m\pi}{L} \cos \frac{m\pi}{L}\, y \sinh \frac{m\pi}{L}\, W$

$$- \sum_{n=1}^{\infty} A_{0n} \frac{n\pi}{L} \cos \frac{n\pi}{L}\, x \sinh \frac{n\pi}{L}\, W$$

$$- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2}\, A_{mn} \cos \frac{n\pi}{L}\, x \cos \frac{m\pi}{L}\, y \sinh \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2}\, W$$

But $u_z(x,y,0) \;=\; 4 \cos \frac{3\pi}{L}\, x \cos \frac{4\pi}{L}\, y$

Comparing coefficients $A_{mn} \,=\, 0$ for $n \,\neq\, 3$ or $m \,\neq\, 4$

For $n \,=\, 3$ and $m \,=\, 4$ we have $4 \,=\, -\frac{5\pi}{L} A_{43} \sinh \frac{5\pi}{L}\, W$

$$A_{43} \;=\; -\frac{4}{\frac{5\pi}{L} \sinh \frac{5\pi}{L}\, W}$$

Note that A_{00} is NOT specified.

$$\boxed{u(x,\, y,\, z) \;=\; A_{00} \,-\, \frac{4}{\frac{5\pi}{L} \sinh \frac{5\pi}{L}\, W} \cos \frac{3\pi}{L}\, x \cos \frac{4\pi}{L}\, y \cosh \frac{5\pi}{L}\,(z \,-\, W)}$$

5. $u_t = \dfrac{1}{r}(r\,u_r)_r + \dfrac{1}{r^2}u_{\theta\theta} + u_{zz}$

$$\text{BC: } u(r,\theta,0) = 0$$

$$u(r,\theta,H) = 0$$

$$u(a,\theta,z) = 0$$

$$u(r,\theta,z,t) = R(r)\Theta(\theta)Z(z)T(t)$$

$$R\Theta ZT' = \frac{1}{r}\Theta\,Z\,T\,(r\,R')' + \frac{1}{r^2}RZT\Theta'' + R\Theta TZ''$$

$$\frac{T'}{T} = \frac{\frac{1}{r}(r\,R')'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z}$$

$$\frac{Z''}{Z} = \frac{T'}{T} - \frac{\frac{1}{r}(r\,R')'}{R} - \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda$$

$$\boxed{\begin{array}{l} Z'' + \lambda Z = 0 \\ BC:\ Z(0) = 0 \\ Z(H) = 0 \end{array}}$$

$\Downarrow$

$$Z_n = \sin\frac{n\pi}{H}z$$

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2 \qquad n = 1, 2, \cdots$$

$$\frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{T'}{T} - \frac{1}{r}\frac{(r\,R')'}{R} + \left(\frac{n\pi}{H}\right)^2$$

$$\frac{\Theta''}{\Theta} = \frac{T'}{T}r^2 - \frac{r(r\,R')'}{R} + \left(\frac{n\pi}{H}\right)^2 r^2 = -\mu$$

$$\boxed{\begin{array}{l} \Theta'' + \mu\Theta = 0 \\ BC:\ \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{array}}$$

$\Downarrow$

$$\Theta_m = \begin{cases} \sin m\,\theta \\ \\ \cos m\,\theta \end{cases}$$

$$\mu_m = m^2 \qquad m = 0, 1, 2, \cdots$$

$$\frac{T'}{T} r^2 = \frac{r(r\,R')'}{R} - \left(\frac{n\pi}{H}\right)^2 r^2 - m^2$$

$$\frac{T'}{T} = \frac{\frac{1}{r}(r\,R')'}{R} - \left(\frac{n\pi}{H}\right)^2 - \frac{m^2}{r^2} = -\nu$$

$$\boxed{T' + \nu T = 0}$$

$$\frac{\frac{1}{r}(r\,R')'}{R} = -\nu + \left(\frac{n\pi}{H}\right)^2 + \frac{m^2}{r^2}$$

$$\frac{r(r\,R')'}{R} = -\nu r^2 + \left(\frac{n\pi}{H}\right)^2 r^2 + m^2$$

$$\boxed{\begin{array}{l} r(r\,R')' - (\nu - \left(\frac{n\pi}{H}\right)^2)\,r^2\,R - m^2\,R = 0 \\ \\ BC: \ |\,R\,(0)\,| < \infty \\ \\ R\,(a) = 0 \end{array}}$$

$$R_{nm\ell} = I_m\left(\sqrt{\nu_\ell - \left(\frac{n\pi}{H}\right)^2}\, r\right)$$

This solution satisfies the boundedness at the origin. The eigenvalues ν_ℓ can be found by using the second boundary condition:

$$I_m\left(\sqrt{\nu_\ell - \left(\frac{n\pi}{H}\right)^2}\, a\right) = 0$$

Since the function $I_m(x)$ vanishes only at zero for any $m = 1, 2, \cdots$ (I_0 is never zero) then there is only one ν (for any n) satisfying

$$\sqrt{\nu - \left(\frac{n\pi}{H}\right)^2}\, a = 0 \qquad m = 1, 2, \cdots$$

$$\nu = \left(\frac{n\pi}{H}\right)^2$$

The solution fot T is $T_{nm} = e^{-\left(\frac{n\pi}{H}\right)^2 t}$

The solution for R is $I_m(0 \cdot r)$ which is identically zero. This means that $u(r, \theta, z, t) = 0$. Physically, this is NOT surprising, since the problem has NO sources (homogeneous boundary conditions and homogeneous PDE).

7.7 Laplace's equation in a sphere

Problems

1. Solve Laplace's equation on the sphere

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{\cot\theta}{r^2}u_\theta + \frac{1}{r^2\sin^2\theta}u_{\varphi\varphi} = 0, \qquad 0 \le r < a,\ 0 < \theta < \pi,\ 0 < \varphi < 2\pi,$$

subject to the boundary condition

$$u_r(a, \theta, \varphi) = f(\theta).$$

2. Solve Laplace's equation on the half sphere

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{\cot\theta}{r^2}u_\theta + \frac{1}{r^2\sin^2\theta}u_{\varphi\varphi} = 0, \qquad 0 \le r < a,\ 0 < \theta < \pi,\ 0 < \varphi < \pi,$$

subject to the boundary conditions

$$u(a, \theta, \varphi) = f(\theta, \varphi),$$

$$u(r, \theta, 0) = u(r, \theta, \pi) = 0.$$

1. $$u(r,\,\theta,\,\varphi) \,=\, \sum_{n=0}^{\infty} A_{n\,0}\, r^n\, P_n(\cos\,\varphi) + \sum_{n=0}^{\infty}\,\sum_{m=1}^{\infty} r^n\, P_n^m(\cos\,\varphi)\,(A_{nm}\,\cos\,m\varphi \,+\, B_{nm}\,\sin\,m\,\varphi) \tag{7.7.37}$$

$$u_r(a,\,\theta,\,\varphi) \,=\, f(\theta) \,= \,=\, \sum_{n=0}^{\infty} n\, A_{n0}\, a^{n-1}\, P_n(\cos\,\theta) + \sum_{n=0}^{\infty}\,\sum_{m=1}^{\infty} n\;a^{n-1}\, P_n^m(\cos\,\theta)\,(A_{nm}\,\cos\,m\varphi \,+\, B_{nm}\,\sin\,m\,\varphi)$$

$$A_{n0}\, n\, a^{n-1} \,=\, \frac{\int_0^{\pi}\, f(\theta)\, P_n(\cos\,\theta)\,\sin\,\theta\, d\theta}{\int_0^{\pi}\, P_n^2(\cos\,\theta)\,\sin\,\theta\, d\theta}$$

$$n\, a^{n-1}\, A_{nm} \,=\, \frac{\int_0^{\pi}\,\int_0^{2\pi}\, f(\theta)\, P_n^m(\cos\,\theta)\,\cos\,m\,\varphi\,\sin\,\theta\, d\varphi\, d\theta}{\int_0^{\pi}\,\int_0^{2\pi}\,[P_n^m(\cos\,\theta)\,\cos\,m\,\varphi]^2\,\sin\,\theta\, d\varphi\, d\theta}$$

$$n\, a^{n-1}\, B_{nm} \,=\, \frac{\int_0^{\pi}\,\int_0^{2\pi}\, f(\theta)\, P_n^m(\cos\,\theta)\,\sin\,m\,\varphi\,\sin\,\theta\, d\varphi\, d\theta}{\int_0^{\pi}\,\int_0^{2\pi}\,[P_n^m(\cos\,\theta)\,\sin\,m\,\varphi]^2\,\sin\,\theta\, d\varphi\, d\theta}$$

$$A_{nm} \,=\, \frac{\int_0^{\pi}\,\int_0^{2\pi}\, f(\theta)\, P_n^m(\cos\,\theta)\,\cos\,m\,\varphi\,\sin\,\theta\, d\varphi\, d\theta}{n\, a^{n-1}\,\int_0^{\pi}\,\int_0^{2\pi}\,[P_n^m(\cos\,\theta)\,\cos\,m\,\varphi]^2\,\sin\,\theta\, d\varphi\, d\theta}$$

$$B_{nm} \,=\, \frac{\int_0^{\pi}\,\int_0^{2\pi}\, f(\theta)\, P_n^m(\cos\,\theta)\,\sin\,m\,\varphi\,\sin\,\theta\, d\varphi\, d\theta}{n\, a^{n-1}\,\int_0^{\pi}\,\int_0^{2\pi}\,[P_n^m(\cos\,\theta)\,\sin\,m\,\varphi]^2\,\sin\,\theta\, d\varphi\, d\theta}$$

2. $\varphi'' + \mu\varphi = 0$

$$\varphi(0) = \varphi(\pi) = 0$$

$$\Downarrow$$

$$\mu_m = m^2$$

$$\varphi_m = \sin m\varphi$$

$$m = 1, 2, \cdots$$

$$\Rightarrow \quad u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} r^n \underbrace{P_n^m(\cos\theta)}_{\Theta \text{ equation}} A_{nm} \sin m\varphi$$

$$\uparrow$$

R equation

$$u(a, \theta, \varphi) = f(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a^n A_{nm} P_n^m(\cos\theta) \sin m\varphi$$

$$a^n A_{nm} = \frac{\int_0^\pi \int_0^\pi f(\theta, \varphi) P_n^m(\cos\theta) \sin m\varphi \underbrace{\sin\theta \, d\theta \, d\varphi}_{\text{area elem.}}}{\int_0^\pi \int_0^\pi (P_n^m(\cos\theta))^2 \sin^2 m\varphi \sin\theta \, d\theta \, d\varphi}$$

$$A_{nm} = \frac{\int_0^\pi \int_0^\pi f(\theta, \varphi) P_n^m(\cos\theta) \sin m\varphi \underbrace{\sin\theta \, d\theta \, d\varphi}_{\text{area elem.}}}{a^n \int_0^\pi \int_0^\pi (P_n^m(\cos\theta))^2 \sin^2 m\varphi \sin\theta \, d\theta \, d\varphi}$$

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} r^n P_n^m(\cos\theta) A_{nm} \sin m\varphi$$

3. The equation becomes

$$u_{\theta\theta} + \cot\theta u_\theta + \frac{1}{\sin^2\theta} u_{\varphi\varphi} = 0, \quad 0 < \theta < \pi, \ 0 < \varphi < 2\pi$$

Using separation of variables

$$u(\theta, \varphi) = \Theta(\theta)\Phi(\varphi)$$

$$\Theta''\Phi + \cot\theta\Theta'\Phi + \frac{1}{\sin^2\theta}\Theta\Phi'' = 0$$

Divide by $\Phi\Theta$ and multiply by $\sin^2\theta$ we have

$$\sin^2\theta\frac{\Theta''}{\Theta} + \cos\theta\sin\theta\frac{\Theta'}{\Theta} = -\frac{\Phi''}{\Phi} = \mu$$

Thus

$$\Phi'' + \mu\Phi = 0$$

$$\sin^2\theta\Theta'' + \sin\theta\cos\theta\Theta' - \mu\Theta = 0$$

Because of periodicity, the Φ equation has solutions

$$\Phi_m = \begin{cases} \sin \quad m\varphi & \qquad m = 1, 2, \dots \\ \\ \cos \quad m\varphi & \end{cases}$$

$$\Phi_0 = 1$$

$$\mu_m = m^2 \qquad m = 0, 1, 2, \dots$$

Substituting these $\mu's$ in the Θ equation, we get (7.7.21) with $\alpha_1 = 0$. The solution of the Θ equation is thus given by (7.7.27) - (7.7.28) with $\alpha_1 = 0$.

8 Separation of Variables-Nonhomogeneous Problems

8.1 Inhomogeneous Boundary Conditions

Problems

1. For each of the following problems obtain the function $w(x,t)$ that satisfies the boundary conditions and obtain the PDE

a.

$$\begin{aligned} u_t(x,t) &= ku_{xx}(x,t) + x, \qquad\qquad 0 < x < L \\ u_x(0,t) &= 1, \\ u(L,t) &= t. \end{aligned}$$

b.

$$\begin{aligned} u_t(x,t) &= ku_{xx}(x,t) + x, \qquad\qquad 0 < x < L \\ u(0,t) &= 1, \\ u_x(L,t) &= 1. \end{aligned}$$

c.

$$\begin{aligned} u_t(x,t) &= ku_{xx}(x,t) + x, \qquad\qquad 0 < x < L \\ u_x(0,t) &= t, \\ u_x(L,t) &= t^2. \end{aligned}$$

2. Same as problem 1 for the wave equation

$$u_{tt} - c^2 u_{xx} = xt, \qquad\qquad 0 < x < L$$

subject to each of the boundary conditions

a.

$$u(0,t) = 1 \qquad\qquad u(L,t) = t$$

b.

$$u_x(0,t) = t \qquad\qquad u_x(L,t) = t^2$$

c.

$$u(0,t) = 0 \qquad\qquad u_x(L,t) = t$$

d.

$$u_x(0,t) = 0 \qquad\qquad u_x(L,t) = 1$$

1a. $u_x(0, t) = 1$

$u(L, t) = t$

$w(x, t) = A(t)x + B(t)$

$1 = w_x(0, t) = A(t) \quad \Rightarrow \quad A(t) = 1$

$t = w(L, t) = A(t)L + B(t) \quad \Rightarrow \quad B(t) = t - L$

$\Rightarrow \quad \underline{w(x, t) = x + t - L}$

$w_t = 1$
$w_{xx} = 0$
$v = u - w \quad \Rightarrow \quad u = v + w$

$\Rightarrow \quad \underline{v_t + 1 = kv_{xx} + x}$

b. $w = Ax + B$
$1 = w(0, t) = B(t)$
$1 = w_x(L, t) = A(t)$
$\underline{w = x + 1}$
$w_t = w_{xx} = 0$

$\underline{v_t = kv_{xx} + x}$

c. $w_x(0,\, t) \,=\, t$

$w_x(L,\, t) \,=\, t^2$

try $\quad w \,=\, A(t)\, x \,+\, B$

$w_x \,=\, A(t)$ and we can <u>not</u> satisfy the 2 conditions.

try $\quad w \,=\, A(t)\, x^2 \,+\, B(t)\, x$

$w_x \,=\, 2A(t)\, x \,+\, B(t)$

$t \,=\, w_x\,(0,\, t) \,=\, B(t)$

$t^2 \,=\, w_x(L,\, t) \,=\, 2A(t)L \,+\, \underbrace{B(t)}_{=t} \quad \Rightarrow \quad A(t) \,=\, \frac{t^2 - t}{2L}$

$$\underline{w \,=\, \frac{t^2 \,-\, t}{2L}\, x^2 \,+\, tx}$$

$$w_t \,=\, \frac{2t \,-\, 1}{2L}\, x^2 \,+\, x$$

$$w_{xx} \,=\, \frac{t^2 \,-\, t}{L}$$

$$v_t \,+\, \left(\tfrac{2t-1}{2L}\, x^2 \,+\, x\right) \,=\, k\,(v_{xx} \,+\, \tfrac{t^2 - t}{L}) \,+\, x$$

$$v_t \,=\, kv_{xx} \,-\, \frac{2t \,-\, 1}{2L}\, x^2 \,-\, x \,+\, k\,\frac{t^2 \,-\, t}{L} \,+\, x$$

$$\underline{v_t \,=\, kv_{xx} \,-\, \frac{2t \,-\, 1}{2L}\, x^2 \,+\, k\,\frac{t^2 \,-\, t}{L}}$$

2. $u_{tt} - c^2 u_{xx} = xt$

a. $w(x, t) = \frac{t-1}{L} x + 1$

$w_{tt} = 0 \qquad w_{xx} = 0$

$\underline{v_{tt} - c^2 v_{xx} = xt}$

b. $w = \frac{t^2 - t}{2L} x^2 + tx$ as in 1c

$w_{tt} = \frac{1}{L} x^2$

$w_{xx} = \frac{t^2 - t}{L}$

$u = v + w$

$v_{tt} + \frac{1}{L} x^2 - c^2 \left(v_{xx} + \frac{t^2 - t}{L} \right) = xt$

$\underline{v_{tt} - c^2 v_{xx} = -\frac{1}{L} x^2 + c^2 \frac{t^2 - t}{L} + xt}$

c. $w(0, t) = 0 \qquad w_x(L, t) = t$

$w = Ax + B \qquad w_x = A$

$B = 0 \qquad A = t$

$\underline{w = tx}$

$w_{tt} = w_{xx} = 0$

$\underline{v_{tt} - c^2 v_{xx} = xt}$

d. $w_x(0,\, t) \,=\, 0$

$w_x(L,\, t) \,=\, 1$

Try $\quad w \,=\, A(t)\,x^2 \,+\, B(t)\,x \quad$ as in 1c

$\quad w_x \,=\, 2A(t)x \,+\, B(t)$

$$0 \,=\, B(t) \qquad 1 \,=\, 2A(t)L \,+\, \underbrace{B(t)}_{=0}$$

$$A(t) \,=\, \frac{1}{2L}$$

$$\underline{w \,=\, \frac{1}{2L}\,x^2}$$

$w_{tt} \,=\, 0 \qquad w_{xx} \,=\, \frac{1}{L}$

$v \,=\, u \,+\, w$

$$v_{tt} \,-\, c^2\left(v_{xx} \,+\, \tfrac{1}{L}\right) \,=\, x\,t$$

$$\underline{v_{tt} \,-\, c^2\,v_{xx} \,=\, +\frac{c^2}{L} \,+\, x\,t}$$

8.2 Method of Eigenfunction Expansions

Problems

1. Solve the heat equation

$$u_t = ku_{xx} + x, \qquad 0 < x < L$$

subject to the initial condition

$$u(x, 0) = x(L - x)$$

and each of the boundary conditions

a.

$$u_x(0, t) = 1,$$
$$u(L, t) = t.$$

b.

$$u(0, t) = 1,$$
$$u_x(L, t) = 1.$$

c.

$$u_x(0, t) = t,$$
$$u_x(L, t) = t^2.$$

2. Solve the heat equation

$$u_t = u_{xx} + e^{-t}, \qquad 0 < x < \pi, \quad t > 0,$$

subject to the initial condition

$$u(x, 0) = \cos 2x, \qquad 0 < x < \pi,$$

and the boundary condition

$$u_x(0, t) = u_x(\pi, t) = 0.$$

1. $u_t = k u_{xx} + x$

$u(x, 0) = x(L - x)$

a. $u_x(0, t) = 1$

$$\Rightarrow \quad w = x + t - L$$

$u(L, t) = t$

Solve $\quad v_t = k v_{xx} - 1 + x \qquad$ (see 1a last section)

$v_x(0, t) = 0 \qquad v(x, 0) = x(L - x) - (x - L) = (x + 1)(L - x)$

$v(L, t) = 0$

eigenvalues: $\quad \left[\left(n - \frac{1}{2}\right) \frac{\pi}{L}\right]^2 \qquad n = 1, 2, \cdots$

eigenfunctions : $\quad \cos\left(n - \frac{1}{2}\right) \frac{\pi}{L} x \qquad n = 1, 2, \cdots$

$$\boxed{v = \sum_{n=1}^{\infty} v_n(t) \cos\left(n - \frac{1}{2}\right) \frac{\pi}{L} x}$$

$$-1 + x = \sum_{n=1}^{\infty} s_n \cos\left(n - \frac{1}{2}\right) \frac{\pi}{L} x \quad \Rightarrow \quad \boxed{s_n = \frac{\int_0^L (-1 + x) \cos\left(n - \frac{1}{2}\right) \frac{\pi}{L} x \, dx}{\int_0^L \cos^2\left(n - \frac{1}{2}\right) \frac{\pi}{L} x \, dx}}$$

$$\sum_{n=1}^{\infty} \dot{v}_n(t) \cos\left(n - \frac{1}{2}\right) \frac{\pi}{L} x = k \sum_{n=1}^{\infty} \left\{ - \left[\left(n - \frac{1}{2}\right) \frac{\pi}{L}\right]^2 \right\} v_n \cos\left(n - \frac{1}{2}\right) \frac{\pi}{L} x$$

$$+ \sum_{n=1}^{\infty} s_n \cos\left(n - \frac{1}{2}\right) \frac{\pi}{L} x$$

Compare coefficients

$\dot{v}_n(t) + k \left(\left(n - \frac{1}{2}\right) \frac{\pi}{L}\right)^2 v_n = s_n$

$$\boxed{v_n = v_n(0) e^{-[\left(n - \frac{1}{2}\right) \frac{\pi}{L}]^2 k t} + s_n \underbrace{\int_0^t e^{-[\left(n - \frac{1}{2}\right) \frac{\pi}{L}]^2 k (t - \tau)} \, d\tau}_{\text{see}(8.\ 2.\ 39)}}$$

$v_n(0) =$ coefficients of expansion of $(1 + x)(L - x)$

$$\boxed{v_n(0) = \frac{\int_0^L (1 + x)(L - x) \cos\left(n - \frac{1}{2}\right) \frac{\pi}{L} x \, dx}{\int_0^L \cos^2\left(n - \frac{1}{2}\right) \frac{\pi}{L} x \, dx}} \qquad \boxed{u = v + w}$$

1b. $u(0,\, t) \,=\, u_x\,(L,\, t) \,=\, 1$

$\Rightarrow w \,=\, x \,+\, 1$

$v_t \,=\, k\, v_{xx} \,+\, x$

$v\,(0,\, t) \,=\, 0$

$v_x\,(L,\, t) \,=\, 0$

$v\,(x,\, 0) \,=\, x\,(L \,-\, x) \,-\, (x \,+\, 1)$

eigenvalues: $\left[\left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\right]^2 \qquad n \,=\, 1,\, 2,\, \cdots$

eigenfunctions: $\sin\,\left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\, x \qquad n \,=\, 1,\, 2,\, \cdots$

$$v \,=\, \sum_{n=1}^{\infty} v_n(t)\, \sin\,\left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\, x$$

$$x \,=\, \sum_{n=1}^{\infty} s_n\, \sin\,\left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\, x \qquad s_n \,=\, \frac{\int_0^L\, x\, \sin\,\left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\, x\, dx}{\int_0^L\, \sin^2\,\left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\, x\, dx}$$

$$\dot{v}_n \,+\, k\, \left[\left(n \,-\, \tfrac{1}{2}\right)\, \tfrac{\pi}{L}\right]^2\, v_n \,=\, s_n$$

$$v_n(t) \,=\, v_n(0)\, e^{-[\left(n - \frac{1}{2}\right)\, \frac{\pi}{L}]^2\, kt} \,+\, s_n\, \frac{1 - e^{-[\left(n - \frac{1}{2}\right)\, \frac{\pi}{L}]^2 t}}{[\left(n - \frac{1}{2}\right)\, \frac{\pi}{L}]^2}$$

$$v_n(0) \,=\, \frac{\int_0^L [x\,(L \,-\, x) \,-\, (x \,+\, 1)]\, \sin\,\left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\, x\, dx}{\int_0^L\, \sin^2\,\left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\, x\, dx}$$

$\uparrow$

Coefficients of expansion of initial condition for v

$\underline{u \,=\, v \,+\, w}$

1c. $u_x(0, t) = t$

$u_x(L, t) = t^2$

$w = \frac{t^2 - t}{2L} x^2 + tx$

$$v_t = kv_{xx} - \underbrace{\frac{2t - 1}{2L} x^2 - x + k \frac{t^2 - t}{L} + x}_{\text{this gives } s_n(t)}$$

$v_x(0, t) = v_x(L, t) = 0 \qquad \Rightarrow \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 0, 1, 2, \cdots$

$X_n(x) = \cos \frac{n\pi}{L} x \quad n = 0, 1, 2, \cdots$

$$v(x, 0) = x(L - x) - \underbrace{w(x, 0)}_{=0} = x(L - x)$$

$$s_n(t) = \frac{\int_0^L \left\{-\frac{2t-1}{2L} x^2 + k \frac{t^2 - t}{L}\right\} \cos \frac{n\pi}{L} x \, dx}{\int_0^L \cos^2 \frac{n\pi}{L} x \, dx}$$

$$v_n(t) = v_n(0) e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \int_0^t s_n(\tau) e^{-k\left(\frac{n\pi}{L}\right)^2 (t-\tau)} d\tau$$

$$v(x, t) = \frac{1}{2} v_0(t) + \sum_{n=1}^{\infty} v_n(t) \cos \frac{n\pi}{L} x$$

$$v_n(0) = \frac{\int_0^L x(L - x) \cos \frac{n\pi}{L} x \, dx}{\int_0^L \cos^2 \frac{n\pi}{L} x \, dx}$$

$$u = v + \frac{t^2 - t}{2L} x^2 + tx$$

2. $u_t = u_{xx} + e^{-t} \qquad 0 < x < \pi\,, \quad t > 0$

$u\,(x,\,0) = \cos 2x \qquad 0 < x < \pi$

$u_x\,(0,\,t) = u_x\,(\pi,\,t) = 0$

Since the boundary conditions are homogeneous we can immediately expand $u(x,\,t)$, the right hand side and the initial temperature distribution in terms of the eigenfunctions. These eigenfunctions are

$\phi_n = \cos nx$

$\lambda = n^2$

$n = 0,\,1,\,2,\,\ldots$

$$u(x,\,t) = \frac{1}{2}u_0(t) + \sum_{n=1}^{\infty} u_n(t)\,\cos nx$$

$$u(x,\,0) = \frac{1}{2}u_0(0) + \sum_{n=1}^{\infty} u_n(0)\,\cos nx = \underbrace{\cos 2x}_{\text{by initial condition}}$$

$$\Rightarrow \begin{cases} u_n(0) = 0 & n \neq 2 \\ u_2(0) = 1 & \end{cases}$$

$$e^{-t} = \sum_{n=1}^{\infty} s_n(t)\,\cos nx + \frac{1}{2}\,s_0(t)$$

$$s_n(t) = \frac{\int_0^\pi e^{-t}\,\cos nx\,dx}{\int_0^\pi \cos^2 nx\,dx} = \frac{e^{-t}\int_0^\pi \cos nx\,dx}{\int_0^\pi \cos^2 nx\,dx}$$

for $n \neq 0$ the numerator is zero !!

For $n = 0$ both integrals yields the same value, thus

$s_0(t) = e^{-t}$

$s_n\,(t) = 0\,, \quad n \neq 0$

Now substitute u_t, u_{xx} from the expansions for u:

$$\frac{1}{2}\dot{u}_0(t) + \sum_{n=1}^{\infty} \dot{u}_n(t)\,\cos nx = \sum_{n=1}^{\infty} (-n^2)\,u_n(t)\,\cos nx + \frac{1}{2}s_0(t) + \sum_{n=1}^{\infty} s_n(t)\,\cos nx$$

For $n = 0$ $\qquad \frac{1}{2}\,\dot{u}_0(t) = \frac{1}{2}\,e^{-t} = \frac{1}{2}\,s_0(t)$

$n \neq 0$ $\qquad \dot{u}_n + n^2\,u_n = 0$

Solve the ODES

$$u_n = C_n\,e^{-n^2 t} \qquad \begin{array}{l} u_n(0) = 0 \qquad n \neq 2 \quad \Rightarrow \quad C_n = 0 \\ u_2(0) = 1 \qquad\qquad\quad \Rightarrow \quad C_2 = 1 \end{array}$$

$$u_0 = -\,e^{-t} + C_0 \qquad u_0(0) = 0 \quad \Rightarrow \quad C_0 - 1 = 0 \quad \Rightarrow \quad C_0 = 1$$

$$\boxed{u(x,\,t) = 1 - e^{-t} + e^{-4t}\cos 2x}$$

8.3 Forced Vibrations

Problems

1. Consider a vibrating string with time dependent forcing

$$u_{tt} - c^2 u_{xx} = S(x,t), \qquad 0 < x < L$$

subject to the initial conditions

$$u(x,0) = f(x),$$
$$u_t(x,0) = 0,$$

and the boundary conditions

$$u(0,t) = u(L,t) = 0.$$

a. Solve the initial value problem.

b. Solve the initial value problem if $S(x,t) = \cos \omega t$. For what values of ω does resonance occur?

2. Consider the following damped wave equation

$$u_{tt} - c^2 u_{xx} + \beta u_t = \cos \omega t, \qquad 0 < x < \pi,$$

subject to the initial conditions

$$u(x,0) = f(x),$$
$$u_t(x,0) = 0,$$

and the boundary conditions

$$u(0,t) = u(\pi,t) = 0.$$

Solve the problem if β is small ($0 < \beta < 2c$).

3. Solve the following

$$u_{tt} - c^2 u_{xx} = S(x,t), \qquad 0 < x < L$$

subject to the initial conditions

$$u(x,0) = f(x),$$
$$u_t(x,0) = 0,$$

and each of the following boundary conditions

a.

$$u(0,t) = A(t) \qquad u(L,t) = B(t)$$

b.

$$u(0,t) = 0 \qquad u_x(L,t) = 0$$

c.

$$u_x(0,t) = A(t) \qquad u(L,t) = 0.$$

4. Solve the wave equation

$$u_{tt} - c^2 u_{xx} = xt, \qquad 0 < x < L,$$

subject to the initial conditions

$$u(x,0) = \sin x$$

$$u_t(x,0) = 0$$

and each of the boundary conditions

a.

$$u(0,t) = 1,$$

$$u(L,t) = t.$$

b.

$$u_x(0,t) = t,$$

$$u_x(L,t) = t^2.$$

c.

$$u(0,t) = 0,$$

$$u_x(L,t) = t.$$

d.

$$u_x(0,t) = 0,$$

$$u_x(L,t) = 1.$$

5. Solve the wave equation

$$u_{tt} - u_{xx} = 1, \qquad 0 < x < L,$$

subject to the initial conditions

$$u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

and the boundary conditions

$$u(0,t) = 1,$$

$$u_x(L,t) = B(t).$$

1a. $u_{tt} - c^2 u_{xx} = S(x, t)$

$$u(x, 0) = f(x)$$

$$u_t(x, 0) = 0$$

$$u(0, t) = u(L, t) = 0$$

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t)\, \phi_n(x)$$

$$S(x, t) = \sum_{n=1}^{\infty} s_n(t)\, \phi_n(x)$$

$$\left.\begin{array}{lcl} \phi_n(x) & = & \sin \frac{n\pi}{L} x \\ \lambda_n & = & \left(\frac{n\pi}{L}\right)^2 \end{array}\right\} \qquad n = 1, 2, \cdots$$

$$s_n(t) = \frac{\int_0^L S(x, t) \sin \frac{n\pi}{L} x\, dx}{\int_0^L \sin^2 \frac{n\pi}{L} x\, dx}$$

$$\ddot{u}_n(t) + c^2 \left(\frac{n\pi}{L}\right)^2 u_n(t) = s_n(t)$$

$$u_n(t) = c_1 \cos \frac{n\pi}{L} ct + c_2 \sin \frac{n\pi}{L} ct + \int_0^t s_n(\tau) \frac{\sin c \frac{n\pi}{L} (t - \tau)}{c \frac{n\pi}{L}}\, d\tau$$

$$u_n(0) = c_1 = \frac{\int_0^L f(x) \sin \frac{n\pi}{L} x\, dx}{\int_0^L \sin^2 \frac{n\pi}{L} x\, dx} \text{ since } u(x, 0) = f(x)$$

$$\dot{u}_n(0) = c_2\, c \frac{n\pi}{L} = 0 \quad \text{since} \quad u_t(x, 0) = 0 \quad \Rightarrow \quad c_2 = 0$$

$$u(x, t) = \sum_{n=1}^{\infty} \left\{ c_1 \cos \frac{n\pi}{L} ct + \frac{L}{c\, n\pi} \int_0^t s_n(\tau) \sin c \frac{n\pi}{L} (t - \tau)\, d\tau \right\} \sin \frac{n\pi}{L} x$$

c_1 is given above.

b. If $S = \cos wt$

$$s_n(t) = \frac{\cos wt \int_0^L \sin \frac{n\pi}{L} x\, dx}{\int_0^L \sin^2 \frac{n\pi}{L} x\, dx} = A_n \cos wt$$

$$\text{where } A_n = \frac{\int_0^L \sin \frac{n\pi}{L} x\, dx}{\int_0^L \sin^2 \frac{n\pi}{L} x\, dx}$$

$$u(x,\, t) = \sum_{n=1}^{\infty} \left\{ c_1 \cos \frac{n\,\pi\,c}{L} t + \frac{L}{c\,n\,\pi} A_n \underbrace{\int_0^t \cos w\tau \sin \frac{n\,\pi\,c}{L} (t - \tau)\, d\tau}_{\text{This integral can be computed}} \right\} \sin \frac{n\,\pi}{L} x$$

c. Resonance occurs when

$$w = c\,\frac{n\,\pi}{L} \text{ for any } n = 1,\, 2,\, \cdots$$

2. $u_{tt} - c^2 u_{xx} + \beta u_t = \cos wt$

$$u(x, 0) = f(x)$$

$$u_t(x, 0) = 0$$

$$u(0, t) = u(\pi, t) = 0 \quad \Rightarrow \quad \phi_n = \sin nx \qquad n = 1, 2, \cdots$$

$$\lambda_n = n^2$$

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin nx$$

$$\cos wt = \sum_{n=1}^{\infty} s_n(t) \sin nx$$

$$s_n(t) = \frac{\cos wt \int_0^{\pi} \sin nx\, dx}{\int_0^{\pi} \sin^2 nx\, dx} = A_n \cos wt$$

$$\sum_{n=1}^{\infty} (\ddot{u}_n + c^2 n^2 u_n + \beta \dot{u}_n) \sin nx = \sum_{n=1}^{\infty} s_n(t) \sin nx$$

$$(*)\ \ddot{u}_n + \beta \dot{u}_n + c^2 n^2 u_n = s_n(t) = A_n \cos wt$$

For the homogeneous:

Let $u_n = e^{\mu t}$ $\qquad (\mu^2 + \beta \mu + c^2 n^2) = 0 \qquad \mu = \frac{-\beta \pm \sqrt{\beta^2 - 4c^2 n^2}}{2}$

For $\beta < 2c$, $\beta^2 - 4c^2 n^2 < 0 \quad \Rightarrow \quad$ complex conjugate roots

$$u_n = \left(c_1 \cos \frac{\sqrt{4c^2 n^2 - \beta^2}}{2} t + c_2 \sin \frac{\sqrt{4c^2 n^2 - \beta^2}}{2} \right) e^{-(\beta/2) t}$$

$\uparrow$

Solution for homogeneous.

Because of damping factor $e^{-(\beta/2) t}$ there should not be a problem of resonance. We must find a particular solution for inhomogeneous.

$$u_n^P = B_n \cos wt + C_n \sin wt$$

$$\dot{u}_n = -B_n w \sin wt + C_n w \cos wt$$

$$\ddot{u}_n = -B_n w^2 \cos wt - C_n w^2 \sin wt$$

Substitute in (*) and compare coefficients of cos $w\,t$

$-B_n\,w^2 \;+\; \beta\,C_n\,w \;+\; c^2\,n^2\,B_n \;=\; A_n$
Compare coefficients of sin $w\,t$

$-C_n\,w^2 \;-\; B_n\,w \;+\; C_n \;=\; 0$

$$\downarrow$$

$$B_n \;=\; \frac{C_n\,(1\;-\;w^2)}{w}$$

$\frac{C_n\,(1-w^2)}{w}\,(c^2\,n^2\;-\;w^2)\;+\;\beta\,C_n\,w\;=\;A_n$

$$C_n\left\{\underbrace{\beta\,w\;+\;\frac{c^2\,n^2\;-\;w^2}{w}\,(1\;-\;w^2)}_{=D_n}\right\}\;=\;A_n$$

$C_n\;=\;\frac{A_n}{D_n}$

$B_n\;=\;\frac{A_n\,(1-w^2)}{D_n\,w}$

$\boxed{u_n^P\;=\;\frac{A_n}{D_n}\,\frac{(1-w^2)}{w}\,\cos\,w\,t\;+\;\frac{A_n}{D_n}\,\sin\,w\,t}$ where

$$D_n\;=\;\beta\,w\;+\;\frac{c^2\,n^2\;-\;w^2}{w}\,(1\;-\;w^2)\qquad A_n\;=\;\frac{\int_0^\pi\,\sin\,n\,x\,dx}{\int_0^\pi\,\sin^2\,nx\,dx}$$

Therefore the general solution of the inhomogeneous is

$$\boxed{u_n\;=\;\left(c_1\,\cos\,\frac{\sqrt{4c^2\,n^2-\beta^2}}{2}\,t\;+\;c_2\,\sin\,\frac{\sqrt{4c^2\,n^2-\beta^2}}{2}\,t\right)\,e^{-(\beta/2)\,t)}\;+\;\frac{A_n}{D_n}\,\frac{1-w^2}{w}\,\cos\,w\,t\;+\;\frac{A_n}{D_n}\,\sin\,w\,t}$$

(**) $$\boxed{u(x,\,t)\;=\;\sum_{n=1}^{\infty}\,u_n(t)\,\sin\,nx}$$

$$u_t(x,\,t)\;=\;\sum_{n=1}^{\infty}\,\dot{u}_n(t)\,\sin\,nx$$

$$\dot{u}_n(t)\;=\;(c_1\,\cos\,rt\;+\;c_2\,\sin\,rt)\,\left(-\tfrac{\beta}{2}\right)\,e^{-\frac{\beta}{2}\,t}\;+\;(-rc_1\,\sin\,rt\;+\;rc_2\,\cos\,rt)\,e^{-\beta/2\,t}$$
$$-\,\tfrac{A_n}{D_n}\,(1\;-\;w^2)\,\sin\,w\,t\;+\;\tfrac{A_n}{D_n}\,w\,\cos\,w\,t$$

where $r=\frac{\sqrt{4c^2\,n^2-\beta^2}}{2}$

$$u_t(x,\,0) \;=\; 0 \;\Rightarrow\quad \sum_{n=1}^{\infty}\left\{c_1\left(-\frac{\beta}{2}\right) + rc_2 + \frac{A_n}{D_n}\,w\right\}\sin\,nx \;=\; 0$$

$$\Rightarrow\quad \boxed{-c_1\,\tfrac{\beta}{2} + rc_2 + \tfrac{A_n}{D_n}\,w \;=\; 0} \qquad (\#)$$

$u(x,\,0) \;=\; f(x) \;\Rightarrow\quad u_n(0)$ are Fourier coefficients of f(x)

$$u_n(0) \;=\; \left(c_1 + \frac{A_n}{D_n}\,\frac{1 - w^2}{w}\right) \;=\; \frac{\int_0^{\pi} f(x)\,\sin\,nx\,dx}{\int_0^{\pi}\sin^2\,nx\,dx} \quad\Rightarrow\quad \text{we have } c_1$$

Use c_1 in (#) to get c_2 and the solution is in (**) with u_n at top of page.

3. $u_{tt} - c^2 u_{xx} = S(x\,t)$

$$u(x,\,0) = f(x)$$

$$u_t(x,\,0) = 0$$

a. $u(0,\,t) = A(t)$

$$u(L,\,t) = B(t) \quad \Rightarrow \quad w = \alpha x + \beta$$

$$\beta = A(t)$$

$$\alpha L + \beta = B$$

$$\alpha = \tfrac{B-\beta}{L}$$

$$\boxed{w = \tfrac{B(t)-A(t)}{L}\,x + A(t)}$$

$$w_{xx} = 0 \qquad w_{tt} = \tfrac{\ddot{B}-\ddot{A}}{L}\,x + \ddot{A}$$

$$v = u - w$$

$$u = v + w$$

$$v_{tt} - c^2 v_{xx} = S(x,\,t) - w_{tt} \equiv \hat{S}(x,\,t)$$

$$v(x,\,0) = f(x) - \tfrac{B(0)-A(0)}{L}\,x - A(0) \equiv F(x)$$

$$v_t(x,\,0) = 0 - w_t\,(x,\,0) = 0 - \tfrac{\dot{B}(0)-\dot{A}(0)}{L}\,x - \dot{A}(0) \equiv G(x)$$

$$v(0,\,t) = 0$$

$$v(L,\,t) = 0$$

Solve the homogeneous

$$\lambda_n = \left(\tfrac{n\pi}{L}\right)^2$$

$$n = 1,\,2,\,\cdots$$

$$\phi_n = \sin\,\tfrac{n\pi}{L}\,x$$

$$\boxed{v(x,\,t) = \sum_{n=1}^{\infty} v_n(t)\,\sin\,\frac{n\pi}{L}\,x}$$

$$\hat{S}(x,\,t) = \sum_{n=1}^{\infty} s_n(t)\,\sin\,\frac{n\pi}{L}\,x \quad , \quad \boxed{s_n(t) = \frac{\int_0^L \hat{S}(x,\,t)\,\sin\,\frac{n\pi}{L}\,x\,dx}{\int_0^L \sin^2\,\frac{n\pi}{L}\,x\,dx}}$$

$$\sum_{n=1}^{\infty} (\ddot{v}_n + c^2 \left(\frac{n\pi}{L}\right)^2 v_n)\,\sin\,\frac{n\pi}{L}\,x = \sum_{n=1}^{\infty} s_n\,\sin\,\frac{n\pi}{L}\,x$$

$\ddot{v}_n + \left(\frac{c\,n\,\pi}{L}\right)^2 v_n = s_n$

$v_n(0)$ coefficient of expanding $F(x)$

$\dot{v}_n(0)$ coefficient of expanding $G(x)$

$$\boxed{v_n = \underbrace{c_1}_{\downarrow} \cos \frac{c\,n\,\pi}{L}\, t + \underbrace{c_2}_{\downarrow} \sin \frac{c\,n\,\pi}{L}\, t + \int_0^t s_n(\tau) \frac{\sin \frac{c\,n\,\pi}{L}(t-\tau)}{\frac{c\,n\,\pi}{L}}\, d\tau}$$

$$v_n(0) \qquad \frac{\dot{v}_n(0)}{\frac{c\,n\,\pi}{L}} \qquad \text{(see 8.3.12-13)}$$

$$\boxed{u = v + w}$$

b. $u(0,\, t) \,=\, 0$

$u_x(L,\, t) \,=\, 0 \quad \Rightarrow \qquad$ Homogeneous. b.c.

$\lambda_n \,=\, \left(\left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\right)^2$

$\phi_n \,=\, \sin\, \frac{(n\,-\,\frac{1}{2})\,\pi}{L}\, x$

$n \,=\, 1,\, 2,\, \cdots$

$$u \,=\, \sum_{n=1}^{\infty} u_n(t)\, \sin\, \frac{(n\,-\,\frac{1}{2})\,\pi}{L}\, x$$

$$S \,=\, \sum_{n=1}^{\infty} s_n(t)\, \sin\, \frac{(n\,-\,\frac{1}{2})\,\pi}{L}\, x$$

$$\ddot{u}_n \,+\, c^2\, \left(\frac{\left(n\,-\,\frac{1}{2}\right)\pi}{L}\right)^2 u_n \,=\, s_n$$

$$u_n \,=\, c_1\, \cos\, \frac{\left(n\,-\,\frac{1}{2}\right)\,\pi\, c}{L}\, t + c_2\, \sin\, \frac{\left(n\,-\,\frac{1}{2}\right)\,\pi\, c}{L}\, t + \int_0^t s_n(\tau)\, \frac{\sin\,(n\,-\,1/2)\,\frac{\pi}{L}\, c\,(t\,-\,\tau)}{\left(\left(n\,-\,\frac{1}{2}\right)\,\frac{\pi\, c}{L}\right)}\, d\tau$$

$\downarrow$

$u_n(0)$ coefficients of $f(x)$

$c_2 \,=\, 0$ (since $u_t \,=\, 0$)

c. $u_x(0,\, t) \,=\, A(t)$

$u(L,\, L) \,=\, 0$

$w \,=\, ax \,+\, b$

$w_x(0,\, t) \,=\, A(t) \,=\, a \qquad \Rightarrow \quad a \,=\, A(t)$

$w(L,\, t) \,=\, 0 \qquad = aL \,+\, b \qquad \Rightarrow \quad b \,=\, -aL \quad \Rightarrow b \,=\, -\,A(t)L$

$w \,=\, A(t)x \,-\, A(t)L \,=\, \underline{A(t)\,(x \,-\, L)}$

$$w_{xx} \,=\, 0 \qquad w_{tt} \,=\, \ddot{A}(x \,-\, L)$$

$v \,=\, u \,-\, w$

$u \,=\, v \,+\, w$

$v_{tt} \,-\, c^2\, v_{xx} \,=\, S(x,\, t) \,-\, \ddot{A}(t)\,(x \,-\, L) \,\equiv\, \hat{S}\,(x,\, t)$

$v\,(x,\, 0) \,=\, f(x) \,-\, A(0)(x \,-\, L) \,\equiv\, F(x)$

$v_t(x,\, 0) \,=\, 0 \,-\, \dot{A}(0)\,(x \,-\, L) \,\equiv G(x)$

continue as in b.

4a. $u_{tt} - c^2 u_{xx} = xt$

$u(x, 0) = \sin x$

$u_t(x, 0) = 0$

$$\left.\begin{aligned} u(0, t) &= 1 \\ u(L, t) &= t \end{aligned}\right\} \quad \Rightarrow \quad w(x, t) = \frac{t - 1}{L} x + 1 \,;\; w_t = \frac{x}{L}$$

$$w_{tt} = 0$$

$v_{tt} - c^2 v_{xx} = xt$

$v(0, t) = v(L, t) = 0 \qquad \Rightarrow \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \phi_n = \sin \frac{n\pi}{L} x \qquad n = 1, 2, \cdots$

$v(x, 0) = \sin x - \left(-\frac{x}{L} + 1\right)$

$v_t(x, 0) = 0 - \frac{x}{L}$

$$v(x, t) = \sum_{n=1}^{\infty} v_n(t) \sin \frac{n\pi}{L} x$$

$$xt = \sum_{n=1}^{\infty} s_n(t) \sin \frac{n\pi}{L} x \qquad \Rightarrow \qquad s_n(t) = \frac{\int_0^L xt \sin \frac{n\pi}{L} x \, dx}{\int_0^L \sin^2 \frac{n\pi}{L} x \, dx}$$

$$\sin x + \frac{x}{L} - 1 = \sum_{n=1}^{\infty} v_n(0) \sin \frac{n\pi}{L} x \qquad \Rightarrow \qquad v_n(0) = \frac{\int_0^L (\sin x + \frac{x}{L} - 1) \sin \frac{n\pi}{L} x \, dx}{\int_0^L \sin^2 \frac{n\pi}{L} x \, dx}$$

$$-\frac{x}{L} = \sum_{n=1}^{\infty} \dot{v}_n(0) \sin \frac{n\pi}{L} x \qquad \dot{v}_n(0) = \frac{-\int_0^L \frac{x}{L} \sin \frac{n\pi}{L} x \, dx}{\int_0^L \sin^2 \frac{n\pi}{L} x \, dx}$$

$$\ddot{v}_n + \left(\frac{n\pi}{L}\right)^2 c^2 v_n = s_n(t)$$

$$\rightarrow v_n = c_1 \cos c\sqrt{\lambda_n}\, t + c_2 \sin c\sqrt{\lambda_n}\, t + \int_0^t s_n(\tau) \frac{\sin c\sqrt{\lambda_n}\,(t - \tau)}{c\sqrt{\lambda_n}} \, d\tau$$

$v_n(0) = c_1$

$\dot{v}_n(0) = c_2\, c\sqrt{\lambda_n}$

continue as in 3b.

b. $u_{tt} - c^2 u_{xx} = xt$

$u(x, 0) = \sin x$

$u_t(x, 0) = 0$

$$\left.\begin{aligned} u_x(0, t) &= t \\ u_x(L, t) &= t^2 \end{aligned}\right\} \quad \Rightarrow \quad w(x, t) = \frac{t^2 - t}{2L} x^2 + tx \qquad w_t = \frac{2t - 1}{2L} x^2 + x$$

$$w_{tt} = \frac{x^2}{L} \qquad w_{xx} = \frac{t^2 - t}{L}$$

Thus $\qquad w(x,0) = 0, \qquad w_t(x,0) = x - \dfrac{x^2}{2L}$

Let $v = u - w$

then

$$v_{tt} - c^2 v_{xx} = \underbrace{xt - \frac{x^2}{L} + c^2 \frac{t^2 - t}{L}}_{s(x,t)}$$

$v_x(0, t) = v_x(L, t) = 0 \qquad \Rightarrow \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \phi_n = \cos \frac{n\pi}{L} x \qquad n = 0, 1, 2, \cdots$

$v(x, 0) = \sin x \qquad$ since $w(x,0) = 0$

$v_t(x, 0) = 0 - x + \frac{x^2}{2L}$

$$v(x, t) = \frac{1}{2} v_0(t) + \sum_{n=1}^{\infty} v_n(t) \cos \frac{n\pi}{L} x$$

$$s(x,t) = \frac{1}{2} s_0(t) + \sum_{n=1}^{\infty} s_n(t) \cos \frac{n\pi}{L} x$$

$$\Rightarrow \quad s_n(t) = \frac{1}{L} \int_0^L \left(xt - \frac{x^2}{L} + c^2 \frac{t^2 - t}{L} \right) \cos \frac{n\pi}{L} x \, dx \qquad n = 0, 1, 2, \cdots$$

$$\sin x = \frac{1}{2} v_0(0) + \sum_{n=1}^{\infty} v_n(0) \cos \frac{n\pi}{L} x$$

$$\Rightarrow \quad v_n(0) = \frac{\int_0^L \sin x \cos \frac{n\pi}{L} x \, dx}{\int_0^L \cos^2 \frac{n\pi}{L} x \, dx} \qquad n = 0, 1, 2, \cdots$$

$$-x + \frac{x^2}{2L} = \frac{1}{2} \dot{v}_0(0) + \sum_{n=1}^{\infty} \dot{v}_n(0) \cos \frac{n\pi}{L} x$$

$$\Rightarrow \quad \dot{v}_n(0) = \frac{\int_0^L \left(-x + \frac{x^2}{2L}\right) \cos \frac{n\pi}{L} x \, dx}{\int_0^L \cos^2 \frac{n\pi}{L} x \, dx} \qquad n = 0, 1, 2, \cdots$$

$$\ddot{v}_n + \left(\frac{n\pi}{L}\right)^2 c^2 v_n = s_n(t)$$

$$\ddot{v}_0 = s_0(t)$$

The solution of the ODE for $n = 0$ is obtained by integration twice and using the initial conditions

$$v_0(t) = \int_0^t \left(\int_0^\xi s_0(\tau) d\tau\right) d\xi + v_0(0) + t\,\dot{v}_0(0)$$

$$v_n = C_n \cos c\frac{n\pi}{L} t + D_n \sin c\frac{n\pi}{L} t + \int_0^t s_n(\tau) \frac{\sin c\frac{n\pi}{L}(t - \tau)}{c\frac{n\pi}{L}} d\tau$$

$$v_n(0) = C_n$$

$$\dot{v}_n(0) = D_n c \frac{n\pi}{L}$$

$$\boxed{v_n(t) = v_n(0) \cos \frac{cn\pi}{L} t + \frac{L\dot{v}_n(0)}{cn\pi} \sin \frac{cn\pi}{L} t + \int_0^t s_n(\tau) \frac{\sin \frac{cn\pi}{L}(t - \tau)}{\frac{cn\pi}{L}} d\tau}$$

Now that we have all the coefficients in the expansion of v, recall that $u = v + w$.

c. $u_{tt} - c^2 u_{xx} = xt$

$u(x,\, 0) = \sin x$

$u_t(x,\, 0) = 0$

$$\left.\begin{aligned} u(0,\, t) &= 0 \\ u_x(L,\, t) &= t \end{aligned}\right\} \quad \Rightarrow \quad w(x,\, t) = xt\,; \qquad w_t = x$$

$$w_{tt} = 0 \quad w_{xx} = 0$$
$$w(x,0) = 0 \quad w_t(x,0) = x$$

$v_{tt} - c^2 v_{xx} = xt$

$v(0,\, t) = v_t(L,\, t) = 0$

$\Rightarrow \qquad \lambda_n = \left(\frac{(n-1/2)\,\pi}{L}\right)^2 \qquad \phi_n = \sin \frac{(n-1/2)\,\pi}{L}\, x \qquad n = 1,\, 2,\, \cdots$

$v(x,\, 0) = \sin x$

$v_t(x,\, 0) = -x$

$$v(x,\, t) = \sum_{n=1}^{\infty} v_n(t) \sin \frac{(n-1/2)\,\pi}{L}\, x$$

$$xt = \sum_{n=1}^{\infty} s_n(t) \sin \frac{(n-1/2)\,\pi}{L}\, x \qquad \Rightarrow \qquad s_n(t) = \frac{\int_0^L xt \sin \frac{(n-1/2)\,\pi}{L}\, x\, dx}{\int_0^L \sin^2 \frac{(n-1/2)\,\pi}{L}\, x\, dx}$$

$$v(x,0) = \sin x = \sum_{n=1}^{\infty} v_n(0) \sin \frac{(n-1/2)\,\pi}{L}\, x \qquad \Rightarrow \qquad v_n(0) = \frac{\int_0^L \sin x \sin \frac{(n-1/2)\,\pi}{L}\, x\, dx}{\int_0^L \sin^2 \frac{(n-1/2)\,\pi}{L}\, x\, dx}$$

$$v_t(x,0) = -x = \sum_{n=1}^{\infty} \dot{v}_n(0) \sin \frac{(n-1/2)\,\pi}{L}\, x \qquad \dot{v}_n(0) = \frac{-\int_0^L x \sin \frac{(n-1/2)\,\pi}{L}\, x\, dx}{\int_0^L \sin^2 \frac{(n-1/2)\,\pi}{L}\, x\, dx}$$

$$\ddot{v}_n + \left(\frac{(n-1/2)\,\pi}{L}\right)^2 c^2 v_n = s_n(t)$$

$$\Rightarrow v_n = c_1 \cos c\sqrt{\lambda_n}\, t + c_2 \sin c\sqrt{\lambda_n}\, t + \int_0^t s_n(\tau) \frac{\sin c\sqrt{\lambda_n}\,(t - \tau)}{c\sqrt{\lambda_n}}\, d\tau$$

$v_n(0) = c_1$

$\dot{v}_n(0) = c_2\, c\sqrt{\lambda_n}$

continue as in 3b.

d. $u_{tt} - c^2 u_{xx} = xt$

$u(x, 0) = \sin x$

$u_t(x, 0) = 0$

$$\left.\begin{aligned} u_x(0, t) &= 0 \\ u_x(L, t) &= 1 \end{aligned}\right\} \quad \Rightarrow \quad w(x, t) = \frac{x^2}{2L}; \qquad w_t = 0$$

$$w_{tt} = 0 \qquad w_{xx} = \tfrac{1}{L}$$

Thus $w(x,0) = \dfrac{x^2}{2L}, \qquad w_t(x,0) = 0$

$$v_{tt} - c^2 v_{xx} = \underbrace{xt + \frac{c^2}{L}}_{s(x,t)}$$

$v_x(0, t) = v_x(L, t) = 0 \qquad \Rightarrow \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \phi_n = \cos \frac{n\pi}{L} x \qquad n = 0, 1, 2, \cdots$

$v(x, 0) = \sin x - \frac{x^2}{2L}$

$v_t(x, 0) = 0$

$$v(x, t) = \frac{1}{2} v_0(t) + \sum_{n=1}^{\infty} v_n(t) \cos \frac{n\pi}{L} x$$

$$s(x,t) = \frac{1}{2} s_0(t) + \sum_{n=1}^{\infty} s_n(t) \cos \frac{n\pi}{L} x$$

$$\Rightarrow \quad s_n(t) = \frac{1}{L} \int_0^L \left(xt + \frac{c^2}{L} \right) \cos \frac{n\pi}{L} x \, dx \qquad n = 0, 1, 2, \cdots$$

$$v(x,0) = \sin x - \frac{x^2}{2L} = \frac{1}{2} v_0(0) + \sum_{n=1}^{\infty} v_n(0) \cos \frac{n\pi}{L} x$$

$$\Rightarrow \quad v_n(0) = \frac{1}{L} \int_0^L \left(\sin x - \frac{x^2}{2L} \right) \cos \frac{n\pi}{L} x \, dx \qquad n = 0, 1, 2, \cdots$$

$$v_t(x,0) = 0 = \frac{1}{2} \dot{v}_0(0) + \sum_{n=1}^{\infty} \dot{v}_n(0) \cos \frac{n\pi}{L} x \qquad \dot{v}_n(0) = 0 \qquad n = 0, 1, 2, \cdots$$

$$\ddot{v}_n + \left(\frac{n\pi}{L}\right)^2 c^2 v_n = s_n(t)$$

$$\ddot{v}_0 = s_0(t)$$

The solution of the ODE for $n = 0$ is obtained by integration twice and using the initial conditions

$$v_0(t) \;=\; \int_0^t \left(\int_0^\xi s_0(\tau) d\tau \right) d\xi \;+\; v_0(0)$$

$$v_n \;=\; C_n \cos c \frac{n\pi}{L} t \;+\; D_n \sin c \frac{n\pi}{L} t \;+\; \int_0^t s_n(\tau) \frac{\sin c \frac{n\pi}{L} (t \,-\, \tau)}{c \frac{n\pi}{L}} d\tau$$

$$v_n(0) \;=\; C_n$$

$$\dot{v}_n(0) \;=\; 0 \;=\; D_n \, c \, \tfrac{n\pi}{L} \qquad \Rightarrow D_n \;=\; 0$$

$$\boxed{v_n(t) \;=\; v_n(0) \cos \frac{cn\pi}{L} t \;+\; \int_0^t s_n(\tau) \frac{\sin \frac{cn\pi}{L} (t \,-\, \tau)}{\frac{cn\pi}{L}} d\tau}$$

Now that we have all the coefficients in the expansion of v, recall that $u \;=\; v \;+\; w$.

5. $u_{tt} - u_{xx} = 1$

$u(x,\, 0) = f(x)$

$u_t(x,\, 0) = g(x)$

$$\left.\begin{aligned} u(0,\, t) &= 1 \\ u_x(L,\, t) &= B(t) \end{aligned}\right\} \quad \Rightarrow \quad w(x,\, t) = xB(t) + 1; \qquad w_t = x\dot{B}(t)$$

$$w_{tt} = x\ddot{B}(t) \quad w_{xx} = 0$$
$$w(x,0) = xB(0) + 1 \quad w_t(x,0) = x\dot{B}(0)$$

$v_{tt} - v_{xx} = 1 - x\ddot{B}(t) + 0 \equiv S(x,t)$

$v(x,0) = f(x) - xB(0) - 1 \equiv F(x)$

$v_t(x,0) = g(x) - x\dot{B}(0) \equiv G(x)$

$v(0,\, t) = v_t(L,\, t) = 0$

$$\Rightarrow \qquad \lambda_n = \left(\frac{(n-1/2)\,\pi}{L}\right)^2 \qquad \phi_n = \sin\,\frac{(n-1/2)\,\pi}{L}\,x \qquad n = 1,\, 2,\, \cdots$$

$$v(x,\, t) = \sum_{n=1}^{\infty} v_n(t)\,\sin\,\frac{(n-1/2)\,\pi}{L}\,x$$

$$S(x,t) = \sum_{n=1}^{\infty} s_n(t)\,\sin\,\frac{(n-1/2)\,\pi}{L}\,x$$

$$\Rightarrow \quad s_n(t) = \frac{\int_0^L S(x,t)\,\sin\,\frac{(n-1/2)\,\pi}{L}\,x\,dx}{\int_0^L \sin^2\,\frac{(n-1/2)\,\pi}{L}\,x\,dx}$$

$$F(x) = \sum_{n=1}^{\infty} v_n(0)\,\sin\,\frac{(n-1/2)\,\pi}{L}\,x \qquad \Rightarrow \quad v_n(0) = \frac{\int_0^L F(x)\,\sin\,\frac{(n-1/2)\,\pi}{L}\,x\,dx}{\int_0^L \sin^2\,\frac{(n-1/2)\,\pi}{L}\,x\,dx}$$

$$G(x) = \sum_{n=1}^{\infty} \dot{v}_n(0)\,\sin\,\frac{(n-1/2)\,\pi}{L}\,x \qquad \Rightarrow \dot{v}_n(0) = \frac{\int_0^L G(x)\sin\,\frac{(n-1/2)\,\pi}{L}\,x\,dx}{\int_0^L \sin^2\,\frac{(n-1/2)\,\pi}{L}\,x\,dx}$$

$$\ddot{v}_n + \left(\tfrac{(n-1/2)\,\pi}{L}\right)^2 v_n = s_n(t)$$

$$\Rightarrow v_n = c_1\,\cos\sqrt{\lambda_n}\,t + c_2\,\sin\sqrt{\lambda_n}\,t + \int_0^t s_n(\tau)\,\frac{\sin\,\sqrt{\lambda_n}\,(t-\tau)}{\sqrt{\lambda_n}}\,d\tau$$

$v_n(0) = c_1$

$\dot{v}_n(0) = c_2\,\sqrt{\lambda_n}$

continue as in 3b.

8.4 Poisson's Equation

Problems

1. Solve

$$\nabla^2 u = S(x,y), \qquad 0 < x < L, \quad 0 < y < H,$$

a.

$$u(0,y) = u(L,y) = 0$$
$$u(x,0) = u(x,H) = 0$$

Use a Fourier sine series in y.

b.

$$u(0,y) = 0 \qquad u(L,y) = 1$$
$$u(x,0) = u(x,H) = 0$$

Hint: Do NOT reduce to homogeneous boundary conditions.

c.

$$u_x(0,y) = u_x(L,y) = 0$$
$$u_y(x,0) = u_y(x,H) = 0$$

In what situations are there solutions?

2. Solve the following Poisson's equation

$$\nabla^2 u = e^{2y} \sin x, \qquad 0 < x < \pi, \quad 0 < y < L,$$

$$u(0,y) = u(\pi,y) = 0,$$
$$u(x,0) = 0,$$
$$u(x,L) = f(x).$$

1. a. $\nabla^2 u = s(x, y)$

$$u(0, y) = u(L, y) = 0$$

$$u(x, 0) = u(x, H) = 0 \quad \Rightarrow \quad \sin \tfrac{n\pi}{H} y$$

Use a Fourier sine series in y (we can also use a Fourier sine series in x or a double Fourier sine series, because of the boundary conditions)

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x) \sin \frac{n\pi}{H} y$$

$$S(x, y) = \sum_{n=1}^{\infty} s_n(x) \sin \frac{n\pi}{H} y$$

$$s_n(x) = \frac{\int_0^H s(x, y) \sin \frac{n\pi}{H} y \, dy}{\int_0^H \sin^2 \frac{n\pi}{H} y \, dy}$$

$$\sum_{n=1}^{\infty} \left\{ \underbrace{-\left(\frac{n\pi}{H}\right)^2 u_n \sin \frac{n\pi}{H} y}_{u_{yy}} + \underbrace{\ddot{u}_n \sin \frac{n\pi}{H} y}_{u_{xx}} \right\} = \sum_{n=1}^{\infty} s_n(x) \sin \frac{n\pi}{H} y$$

$$\underline{\ddot{u}_n(x) - \left(\tfrac{n\pi}{H}\right)^2 u_n(x) = s_n(x)}$$

Boundary conditions are coming from $u(0, y) = u(L, y) = 0$

$$\sum_{n=1}^{\infty} u_n(0) \sin \frac{n\pi}{H} y = 0 \quad \Rightarrow \quad \underline{u_n(0) = 0}$$

$$\sum_{n=1}^{\infty} u_n(L) \sin \frac{n\pi}{H} y = 0 \quad \Rightarrow \quad \underline{u_n(L) = 0}$$

$$(*)\ u_n(x) = \frac{\sinh \frac{n\pi}{H}(L - x)}{-\frac{n\pi}{H} \sinh \frac{n\pi}{H} L} \int_0^x s_n(\xi) \sinh \frac{n\pi}{H} \xi \, d\xi + \frac{\sinh \frac{n\pi}{H} x}{-\frac{n\pi}{H} \sinh \frac{n\pi}{H} L} \int_x^L s_n(\xi) \sinh \frac{n\pi}{H} (L - \xi) \, d\xi$$

Let's check by using (*)

$u_n(0) =$ 1^{st} term the integral is zero since limits are same

2^{nd} term the numerator is zero $= \sinh \frac{n\pi}{H} \cdot 0$

$u_n(L) =$ 1^{st} term the numerator $\sinh \frac{n\pi}{H}(L - L) = 0$

2^{nd} term the integral is zero since limits of integration are the same.

$$\dot{u}_n = \frac{-\frac{n\pi}{H}\cosh\frac{n\pi}{H}(L-x)}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}\int_0^x s_n(\xi)\sinh\frac{n\pi}{H}\xi\, d\xi + \frac{\sinh\frac{n\pi}{H}(L-x)}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}\quad \underbrace{s_n(x)\sinh\frac{n\pi}{H}x}_{\text{integrand at upper limit}}$$

$$+\frac{\frac{n\pi}{H}\cosh\frac{n\pi}{H}x}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}\int_x^L s_n(\xi)\sinh\frac{n\pi}{H}(L-\xi)\, d\xi + \frac{\sinh\frac{n\pi}{H}x}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}\underbrace{\left[-s_n(x)\sinh\frac{n\pi}{H}(L-x)\right]}_{\text{integrand at lower limit}}$$

Let's add the second and fourth terms up

$$-\frac{1}{\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}s_n(x)\left\{\underbrace{\sinh\frac{n\pi}{H}(L-x)\sinh\frac{n\pi}{H}x - \sinh\frac{n\pi}{H}x\sinh\frac{n\pi}{H}(L-x)}_{=0}\right\}$$

$$\ddot{u}_n = \frac{\left(-\frac{n\pi}{H}\right)^2\sinh\frac{n\pi}{H}(L-x)}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}\int_0^x s_n(\xi)\sinh\frac{n\pi}{H}\xi\, d\xi + \frac{-\frac{n\pi}{H}\cosh\frac{n\pi}{H}(L-x)}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}s_n(x)\sinh\frac{n\pi}{H}x$$

$$+\frac{\left(\frac{n\pi}{H}\right)^2\sinh\frac{n\pi}{H}x}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}\int_x^L s_n(\xi)\sinh\frac{n\pi}{H}(L-\xi)\, d\xi + \frac{\frac{n\pi}{H}\cosh\frac{n\pi}{H}x}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}\left[-s_n(x)\sinh\frac{n\pi}{H}(L-x)\right]$$

Let's add the second and fourth terms up

$$\frac{s_n(x)}{-\sinh\frac{n\pi}{H}L}\qquad\left\{\underbrace{\sinh\frac{n\pi}{H}x\cosh\frac{n\pi}{H}(L-x) + \cosh\frac{n\pi}{H}x\sinh\frac{n\pi}{H}(L-x)}_{=\sinh\frac{n\pi}{H}(x-(L-x))=\sin h\frac{n\pi}{H}L}\right\} = s_n(x)$$

The integral terms in $\ddot{u}_n$ are exactly $\left(\frac{n\pi}{H}\right)^2 u_n$ and thus the ODE is satisfied.

b. $\nabla^2 u = S(x, y)$

$$u(0, y) = 0 \qquad u(L, y) = 1$$

$$u(x, 0) = u(x, H) = 0 \quad \Rightarrow \quad \sin \tfrac{n\pi}{H} y$$

Use a Fourier sine series in y

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x) \sin \frac{n\pi}{H} y$$

$$S(x, y) = \sum_{n=1}^{\infty} s_n(x) \sin \frac{n\pi}{H} y$$

$$s_n(x) = \frac{\int_0^H S(x, y) \sin \frac{n\pi}{H} y \, dy}{\int_0^H \sin^2 \frac{n\pi}{H} y \, dy}$$

$$\sum_{n=1}^{\infty} \left\{ \underbrace{-\left(\frac{n\pi}{H}\right)^2 u_n \sin \frac{n\pi}{H} y}_{u_{yy}} + \underbrace{\ddot{u}_n \sin \frac{n\pi}{H} y}_{u_{xx}} \right\} = \sum_{n=1}^{\infty} s_n(x) \sin \frac{n\pi}{H} y$$

$$\underline{\ddot{u}_n(x) - \left(\tfrac{n\pi}{H}\right)^2 u_n(x) = s_n(x)}$$

Boundary conditions are coming from $u(0, y) = 0 \qquad u(L, y) = 1$

$$\sum_{n=1}^{\infty} u_n(0) \sin \frac{n\pi}{H} y = 0 \quad \Rightarrow \quad \underline{u_n(0) = 0}$$

$$\sum_{n=1}^{\infty} u_n(L) \sin \frac{n\pi}{H} y = 1 \quad \Rightarrow \quad u_n(L) = \frac{1}{H} \int_0^H 1 \cdot \sin \frac{n\pi}{H} y \, dy$$

$\Rightarrow \quad u_n(L) = \dfrac{4}{n\pi}$ for n odd and 0 for n even. (see (5.8.1)

For n even the solution is as in 1a (since $u_n(L) = 0$)

For n odd, how would the solution change?

Let $\omega_n = \dfrac{4}{n\pi L} x$, then $\ddot{\omega}_n = 0$

Let $v_n = u_n - \omega_n$ then $v_n(0) = 0$ and $v_n(L) = u_n(L) - \omega_n(L) = 0$

and

$$\ddot{v}_n - \left(\tfrac{n\pi}{H}\right)^2 v_n = \underbrace{\left(\frac{n\pi}{H}\right)^2 \frac{4x}{n\pi L} + s_n}_{\text{This is the } s_n \text{ to be used in (*) in 1a}}$$

c. $\nabla^2 u = S(x, y)$

$$u_x(0, y) = 0 \qquad u_x(L, y) = 0 \quad \Rightarrow \quad \cos \tfrac{n\pi}{L} x$$

$$u_y(x, 0) = u_y(x, H) = 0 \quad \Rightarrow \quad \cos \tfrac{m\pi}{H} y$$

Use a double Fourier cosine series

$$u(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_{nm} \cos \frac{n\pi}{L} x \cos \frac{m\pi}{H} y$$

$$S(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} s_{nm} \cos \frac{n\pi}{L} x \cos \frac{m\pi}{H} y$$

$$s_{nm} = \frac{\int_0^H \int_0^L S(x, y) \cos \frac{m\pi}{H} y \cos \frac{n\pi}{L} x \, dx \, dy}{\int_0^H \int_0^L \cos^2 \frac{m\pi}{H} y \cos^2 \frac{n\pi}{L} x \, dx \, dy}$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-u_{nm}) \left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 \right] \cos \frac{n\pi}{L} x \cos \frac{m\pi}{H} y = S(x, y)$$

Thus
$-u_{nm} \left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 \right] = s_{nm}$

Substituing for s_{nm}, we get the unknowns u_{nm}

$$u_{nm} = \frac{\int_0^H \int_0^L S(x, y) \cos \frac{m\pi}{H} y \cos \frac{n\pi}{L} x \, dx \, dy}{\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 \right] \int_0^H \int_0^L \cos^2 \frac{m\pi}{H} y \cos^2 \frac{n\pi}{L} x \, dx \, dy}$$

What if $\lambda_{nm} = 0$? (i.e. $n = m = 0$)
Then we cannot divide by λ_{nm} but in this case we have zero on the left
$\Rightarrow \int_0^H \int_0^L S(x, y) \, dx \, dy = 0$

This is typical of Neumann boundary conditions.

2. $\nabla^2 u = e^{2y} \sin x$

$$u(0, y) = 0 \qquad u(\pi, y) = 0 \quad \Rightarrow \quad \sin nx$$

$$u(x, 0) = 0 \qquad u(x, L) = f(x)$$

Use a Fourier sine series in x

$$u(x, y) = \sum_{n=1}^{\infty} u_n(y) \sin nx$$

$S(x, y)$ is already in a Fourier sine series in x with the coefficients $s_1(y) = e^{2y}$ and all the other coefficients are zero.

$$\sum_{n=1}^{\infty} \left\{ \underbrace{-n^2 u_n \sin nx}_{u_{xx}} + \underbrace{\ddot{u}_n \sin nx}_{u_{yy}} \right\} = \sum_{n=1}^{\infty} s_n(x) \sin nx$$

$$\ddot{u}_n(y) - n^2 u_n(y) = 0 \qquad \text{for } n \neq 1$$

$$\ddot{u}_1(y) - u_1(y) = e^{2y}$$

Boundary conditions are coming from $u(x, 0) = 0 \qquad u(x, L) = f(x)$

$$\sum_{n=1}^{\infty} u_n(0) \sin nx = 0 \quad \Rightarrow \quad \underline{u_n(0) = 0}$$

$$\sum_{n=1}^{\infty} u_n(L) \sin nx = f(x) \quad \Rightarrow \quad u_n(L) = \frac{1}{L} \int_0^L f(x) \sin nx \, dx$$

The solution of the ODEs is

$$u_1(y) = \underbrace{\frac{1}{3} e^{2y}}_{\text{particular solution}} + \alpha_1 \sinh y + \beta_1 \cosh y$$

and

$$u_n(y) = \alpha_n \sinh ny + \beta_n \cosh ny \qquad n \neq 1$$

Since $u_1(0) = 0$ we have $\frac{1}{3} + \beta_1 = 0 \qquad \Rightarrow \beta_1 = -\frac{1}{3}$

Since $u_n(0) = 0$ we have $\beta_n = 0 \qquad n \neq 1$

Using $u_1(L)$ we have $\frac{1}{3} e^{2L} + \alpha_1 \sinh L - \frac{1}{3} \cosh L = u_1(L)$. This gives a value for α_1

$$\alpha_1 = \frac{u_1(L) + \frac{1}{3} \cosh L - \frac{1}{3} e^{2L}}{\sinh L}$$

Using $u_n(L)$ we get a vlaue for α_n

$$\alpha_n = \frac{u_n(L)}{\sinh nL} \qquad n \neq 1$$

Now we can write the solution

$$u(x,y) = \left(\alpha_1 \sinh y - \frac{1}{3}\cosh y + \frac{1}{3}\, e^{2L}\right) \sin x + \sum_{n=2}^{\infty} \left(\frac{u_n(L)}{\sinh nL} \sinh ny\right) \sin nx$$

with α_1 as above.

9 Fourier Transform Solutions of PDEs

9.1 Motivation

9.2 Fourier Transform pair

Problems

1. Show that the Fourier transform is a linear operator, i. e.

$$\mathcal{F}\left(c_1 f(x) + c_2 g(x)\right) = c_1 \mathcal{F}\left(f(x)\right) + c_2 \mathcal{F}\left(g(x)\right)\ .$$

2. If $F(\omega)$ is the Fourier tranform of $f(x)$, show that the inverse Fourier transform of $e^{-i\omega\beta}F(\omega)$ is $f(x-\beta)$. This is known as the shift theorem.

3. Determine the Fourier transform of

$$f(x) = \begin{cases} 0 & |x| > a \\ 1 & |x| < a\,. \end{cases}$$

4. Determine the Fourier transform of

$$f(x) \;=\; \int_0^x \phi(t)dt.$$

5. Prove the scaling theorem

$$\mathcal{F}\left(f(ax)\right) \;=\; \frac{1}{|a|}\,\mathcal{F}\left(f(x)\right).$$

6. If $F(\omega)$ is the Fourier tranform of $f(x)$, prove the translation theorem

$$\mathcal{F}\left(e^{iax}f(x)\right) \;=\; F(\omega - a).$$

1.

$$\mathcal{F}\left(c_1 f(x) + c_2 g(x)\right) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(c_1 f(x) + c_2 g(x)\right)e^{-i\omega x}dx$$

The integral of a sum is the sum of the integrals:

$$=\; c_1 \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx}_{\mathcal{F}(f(x))} + c_2 \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)e^{-i\omega x}dx}_{\mathcal{F}(g(x))}$$

$$=\; c_1\mathcal{F}\left(f(x)\right) \;+\; c_2\mathcal{F}\left(g(x)\right)$$

2.

$$\mathcal{F}^{-1}\left(e^{-i\omega\beta}F(\omega)\right) \;=\; \int_{-\infty}^{\infty} F(\omega)e^{-i\omega\beta}e^{i\omega x}d\omega$$

$$=\; \int_{-\infty}^{\infty} F(\omega)e^{i\omega(x-\beta)}d\omega$$

$$=\; f(x-\beta)$$

3.

$$\mathcal{F}\left(f(x)\right) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

$$=\; \frac{1}{2\pi}\int_{-a}^{a} 1\cdot e^{-i\omega x}dx$$

$$=\; \frac{1}{2\pi}\,\frac{1}{-i\omega}\,e^{-i\omega x}\Big|_{-a}^{a}$$

$$=\; \frac{-1}{2\pi i\omega}\underbrace{\left(e^{-i\omega a} \;-\; e^{i\omega a}\right)}_{-2i\sin\omega a} \;=\; \frac{2i\sin\omega a}{2\pi i\omega}$$

$$=\; \frac{\sin\omega a}{\pi\omega}$$

4. Differentiating

$$f(x) = \int_0^x \phi(t)dt$$

we get

$$\frac{df}{dx} = \phi(x).$$

Now we can either use the definition (9.2.1) or we can use (9.4.2) saying

$$\mathcal{F}\left(\frac{df}{dx}\right) = i\omega\mathcal{F}(f)$$

to get

$$\mathcal{F}(\phi) = i\omega\mathcal{F}\left(\int_0^x \phi(t)dt\right).$$

Therefore dividing by $i\omega$

$$\mathcal{F}\left(\int_0^x \phi(t)dt\right) = \frac{1}{i\omega}\mathcal{F}(\phi).$$

5. Say $a > 0$, then

$$\mathcal{F}\left(f(ax)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x} f(ax)dx$$

can be transformed by the substitution $y = ax$ to (remember $dy = a\,dx$)

$$\mathcal{F}\left(f(ax)\right) = \frac{1}{2\pi a}\int_{-\infty}^{\infty} e^{-i\omega/ay} f(y)dy = \frac{1}{a}F\left(\frac{\omega}{a}\right)$$

If $a < 0$ then the transformation reverses the limits of integration and that will pull a negative sign in front. So we have $-\frac{1}{a}$ multiplying the integral and that's $\frac{1}{|a|}$ in this case.

6.

$$\begin{aligned}
\mathcal{F}\left(e^{iax}f(x)\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iax}f(x)e^{-i\omega x}dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i(\omega-a)x}dx \\
&= F(\omega-a)
\end{aligned}$$

9.3 Heat Equation

Problems

1. Use Fourier transform to solve the heat equation

$$
\begin{aligned}
u_t = u_{xx} + u, \qquad -\infty < x < \infty, \qquad t > 0, \\
u(x,0) = f(x)\,.
\end{aligned}
$$

1.

The Fourier transform of the equation is

$$U_t(\omega, t) \,=\, -\omega^2 U \,+\, U \,=\, (1-\omega^2)U$$

subject to

$$U(\omega, 0) \,=\, F(\omega)$$

Thus

$$U(\omega, t) \,=\, e^t F(\omega) e^{-\omega^2 t}$$

The inverse Fourier transform is

$$u(x, t) \,=\, e^t \int_{-\infty}^{\infty} F(\omega) e^{-\omega^2 t} e^{i\omega x} d\omega$$

since e^t is independent of ω. The integral is the same as was done in class and the solution is (see (9.3.8))

$$\boxed{u(x, t) \,=\, e^t \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(\xi) e^{-\frac{(x-\xi)^2}{4t}} d\xi}$$

9.4 Fourier Transform of Derivatives

Problems

1. Solve the diffusion-convection equation

$$
\begin{gathered}
u_t = k u_{xx} + c u_x, \qquad -\infty < x < \infty\,, \\
u(x,0) = f(x)\,.
\end{gathered}
$$

2. Solve the linearized Korteweg-de Vries equation

$$
\begin{gathered}
u_t = k u_{xxx}\,, \qquad -\infty < x < \infty\,, \\
u(x,0) = f(x)\,.
\end{gathered}
$$

3. Solve Laplace's equation

$$
u_{xx} + u_{yy} = 0\,, \qquad 0 < x < L\,, \qquad -\infty < y < \infty\,,
$$

subject to

$$
\begin{aligned}
u(0,y) &= g_1(y)\,, \\
u(L,y) &= g_2(y)\,.
\end{aligned}
$$

4. Solve the wave equation

$$
\begin{aligned}
&u_{tt} = u_{xx}\,, \qquad -\infty < x < \infty\,, \\
&u(x,0) = 0\,, \\
&u_t(x,0) = g(x)\,.
\end{aligned}
$$

1. Use Fourier transform to solve the heat equation

$$u_t = ku_{xx} + cu_x, \qquad -\infty < x < \infty, \qquad t > 0,$$
$$u(x,0) = f(x)\,.$$

Taking the Fourier transform, we have

$$U_t = -k\omega^2 U + i\omega c U, \qquad t > 0,$$
$$U(\omega,0) = F(\omega)\,.$$

The solution of this initial value problem in t is

$$U(\omega,t) = F(\omega)e^{-(k\omega^2 - ic\omega)t}$$

Now we find the inverse Fourier transform

$$u(x,t) \;=\; \int_{-\infty}^{\infty} F(\omega)e^{-(k\omega^2 - ic\omega)t}e^{i\omega x}d\omega$$

$$u(x,t) \;=\; \int_{-\infty}^{\infty} \underbrace{e^{i\omega ct}F(\omega)}_{H(\omega)}\underbrace{e^{-k\omega^2 t}}_{G(\omega)}e^{i\omega x}d\omega$$

Therefore

$$u(x,t) \;=\; h * g, \qquad \text{using the convolution theorem}$$

where

$$h \;=\; \mathcal{F}^{-1}\left(H(\omega)\right) \;=\; f(x+ct) \qquad \text{see problem 2 in section 9.2}$$

$$g = \mathcal{F}^{-1}\left(G(\omega)\right) \;=\; \sqrt{\frac{\pi}{kt}}\,e^{-\frac{x^2}{4kt}}$$

Therefore

$$u(x,t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi+ct)\sqrt{\frac{\pi}{kt}}\,e^{-\frac{(x-\xi)^2}{4kt}}d\xi$$

2.
The Fourier transform is

$$U_t = (i\omega)^3 \, kU = -ik\omega^3 U$$

The solution of this differential equation with the initial condition $U(\omega, 0) = F(\omega)$ is given by

$$U = F(\omega)\, e^{-ik\omega^3 t}$$

Let

$$G(\omega) = e^{-ik\omega^3 t}$$

then

$$U(\omega, t) = F(\omega)G(\omega)$$

We can now use the convolution theorem. If

$$g(x) = \int_{-\infty}^{\infty} e^{-ik\omega^3 t}\, e^{i\omega x} d\omega = \int_{-\infty}^{\infty} e^{i(k\omega^3 t - \omega x)} d\omega$$

then

$$\boxed{u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi}$$

The question is how to find $g(x)$
Let $k\omega^3 t = s^3/3$ after using symmetry we get

$$g(x) = \int_{-\infty}^{\infty} e^{i(k\omega^3 t - \omega x)} d\omega = 2\int_0^{\infty} \cos(k\omega^3 t - \omega x) d\omega = \frac{2}{(3kt)^{1/3}} \int_0^{\infty} \cos\left(\frac{s^3}{3} - \frac{sx}{(3kt)^{1/3}}\right) ds$$

$$g(x) = \frac{2\pi}{(3kt)^{1/3}}\, A_i\left(\frac{-x}{(3kt)^{1/3}}\right)$$

where $A_i(x)$ is the Airy function (the solution of $y'' - xy = 0$ satisfying $\lim_{x\to\pm\infty} y = 0$ and $y(0) = \dfrac{3^{-2/3}}{\Gamma(2/3)}$.)

The plot of Airy function $Ai(x)$ is given as figure 60.

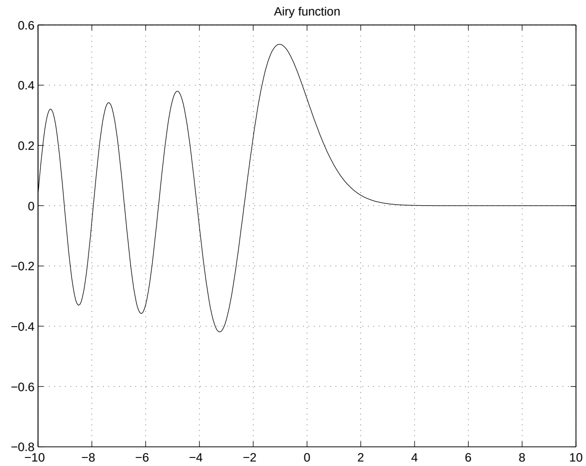


Figure 60: Airy function

3.

$$u_{xx} + u_{yy} = 0\,, \qquad 0 < x < L\,, \qquad -\infty < y < \infty\,,$$

subject to

$$\begin{aligned} u(0,y) &= g_1(y)\,, \\ u(L,y) &= g_2(y)\,. \end{aligned}$$

Use Fourier transform in y

$$U_{xx}(x,\omega) - \omega^2 U(x,\omega) = 0$$

$$U(0,\omega) = G_1(\omega)$$

$$U(L,\omega) = G_2(\omega)$$

The solution is

$$U(x,\omega) = G_1(\omega)\,\frac{\sinh \omega x}{\sinh \omega L} + G_2(\omega)\,\frac{\sinh \omega(L-x)}{\sinh \omega L}$$

Now take the inverse transform.

4. The Fourier transform of the PDE and the initial conditions

$$U_{tt} \;=\; -\,\omega^2 U$$

$$U(\omega,0) \;=\; 0$$

$$U_t(\omega,0) \;=\; G(\omega)$$

The solution is

$$U \;=\; A(\omega)\cos\omega t \;+\; B(\omega)\sin\omega t$$

where $A(\omega) \;=\; 0$ and $B(\omega) \;=\; G(\omega)/\omega$

$$U(\omega,t) \;=\; \frac{G(\omega)}{\omega}\,\sin\omega t$$

Using the inverse transform formula

$$u(x,t) \;=\; \int_{-\infty}^{\infty} G(\omega)\,\frac{\sin\omega t}{\omega}\,e^{i\omega x}\,d\omega$$

This is a convolution of $g(x)$ with the function

$$f(x) = \begin{cases} 0 & |x| > t \\ \pi & |x| < t\,. \end{cases}$$

since (see table of transforms)

$$\mathcal{F}(h(x)) \;=\; \frac{1}{\pi}\,\frac{\sin a\omega}{\omega}$$

for

$$h(x) = \begin{cases} 0 & |x| > a \\ 1 & |x| < a\,. \end{cases}$$

Note that $h(x) \;=\; \pi f(x)$

$$u(x,t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{g(x-\xi)}_{\text{initial condition}} \qquad \underbrace{f(\xi)}_{\pi \quad \text{if } |\xi|<t} \quad d\xi$$

$$\boxed{u(x,t) \;=\; \frac{1}{2}\int_{-t}^{t} g(x-\xi)\,d\xi}$$

9.5 Fourier Sine and Cosine Transforms

Problems

1.
   a. Derive the Fourier cosine transform of $e^{-\alpha x^2}$.
   b. Derive the Fourier sine transform of $e^{-\alpha x^2}$.

2. Determine the inverse cosine transform of $\omega e^{-\omega\alpha}$ (Hint: use differentiation with respect to a parameter)

3. Solve by Fourier sine transform:

$$\begin{aligned} &u_t = k u_{xx}\,, \qquad x > 0\,, \qquad t > 0 \\ &u(0,t) = 1\,, \\ &u(x,0) = f(x)\,. \end{aligned}$$

4. Solve the heat equation

$$\begin{aligned} &u_t = k u_{xx}\,, \qquad x > 0\,, \qquad t > 0 \\ &u_x(0,t) = 0\,, \\ &u(x,0) = f(x)\,. \end{aligned}$$

5. Prove the convolution theorem for the Fourier sine transforms, i.e. (9.5.14) and (9.5.15).

6. Prove the convolution theorem for the Fourier cosine transforms, i.e. (9.5.16).

7.
   a. Derive the Fourier sine transform of $f(x) = 1$.
   b. Derive the Fourier cosine transform of $f(x) = \int_0^x \phi(t)dt$.
   c. Derive the Fourier sine transform of $f(x) = \int_0^x \phi(t)dt$.

8. Determine the inverse sine transform of $\frac{1}{\omega}e^{-\omega\alpha}$ (Hint: use integration with respect to a parameter)

1. a.

$$C\left[e^{-\alpha x^2}\right] = \frac{2}{\pi}\int_0^\infty e^{-\alpha x^2}\cos\omega x\,dx \qquad \text{by definition}$$

Using the symmetry (since the integrand is an even function) we have

$$C\left[e^{-\alpha x^2}\right] = \frac{1}{\pi}\int_{-\infty}^\infty e^{-\alpha x^2}\,e^{i\omega x}\,dx$$

Recall the relationship between the cosine and complex exponentials.

$$C\left[e^{-\alpha x^2}\right] = 2\mathcal{F}\left(e^{-\alpha x^2}\right) = \frac{1}{\sqrt{\pi\alpha}}\,e^{-\frac{\omega^2}{4\alpha}}$$

b.

$$S\left[e^{-\alpha x^2}\right] = \frac{2}{\pi}\int_0^\infty e^{-\alpha x^2}\sin\omega x\,dx \qquad \text{by definition}$$

$$= \frac{1}{\pi i}\int_0^\infty e^{-\alpha x^2+i\omega x}\,dx - \frac{1}{\pi i}\int_0^\infty e^{-\alpha x^2-i\omega x}\,dx$$

Use the transformation $z = -x$ on the second integral.

Thus $dz = -dx$ and $-\alpha x^2 - i\omega x = -\alpha z^2 + i\omega z$.

$$= \frac{1}{\pi i}\int_0^\infty e^{-\alpha x^2+i\omega x}\,dx - \frac{1}{\pi i}\int_0^{-\infty} e^{-\alpha z^2+i\omega z}\,(-dz)$$

Now change the dummy variable of integration z to x and reverse the limits on the second integral

$$= \frac{1}{\pi i}\int_0^\infty e^{-\alpha x^2+i\omega x}\,dx - \frac{1}{\pi i}\int_{-\infty}^0 e^{-\alpha x^2+i\omega x}\,dx$$

Notice that the integrals are similar except for the limits.

2.

$$C^{-1}\left[\omega\, e^{-\alpha\omega}\right] = \int_0^\infty \underbrace{\omega\, e^{-\alpha\omega}}_{-\frac{\partial}{\partial\alpha}(e^{-\omega\alpha})} \cos\omega x d\omega$$

$$= -\frac{\partial}{\partial\alpha}\int_0^\infty e^{-\alpha\omega} \cos\omega x d\omega$$

$$= -\frac{\partial}{\partial\alpha} \underbrace{C\left[e^{-\alpha\omega}\right]}_{=\frac{\alpha}{x^2+\alpha^2} \quad \text{from table}}$$

$$= -\frac{\partial}{\partial\alpha}\frac{\alpha}{x^2+\alpha^2} = -\frac{1\cdot(x^2+\alpha^2)-\alpha\cdot 2\alpha}{(x^2+\alpha^2)^2}$$

$$= -\frac{x^2+\alpha^2-2\alpha^2}{(x^2+\alpha^2)^2} = -\frac{x^2-\alpha^2}{(x^2+\alpha^2)^2}$$

3. Since u is given on the boundary, we use the Fourier sine transform:

$$U_t = k\left[\frac{2}{\pi}\omega u(0,t) - \omega^2 U\right]$$

Substitute the boundary condition, we get

$$U_t = \frac{2k}{\pi}\omega - \omega^2 kU$$

The solution of this ODE is

$$U(\omega,t) = c(\omega)\, e^{-k\omega^2 t} + \frac{2}{\pi\omega}$$

Use the initial condition

$$F(\omega) = U(\omega,0) = c(\omega) + \frac{2}{\pi\omega}$$

Therefore

$$c(\omega) = F(\omega) - \frac{2}{\pi\omega}$$

Plug this c in the solution

$$U(\omega,t) = \left[F(\omega) - \frac{2}{\pi\omega}\right] e^{-k\omega^2 t} + \frac{2}{\pi\omega}$$

This can be written as

$$S[u] = U(\omega,t) = \underbrace{F(\omega)\, e^{-k\omega^2 t}}_{S[f]\cdot C[g]} + \frac{2}{\pi\omega} - \frac{2}{\pi\omega}\, e^{-k\omega^2 t}$$

where

$$G(\omega) = e^{-k\omega^2 t}$$

We now use convolution

$$S[f]\cdot C[g] = \frac{1}{\pi}\int_0^\infty f(\xi)\,[g(x-\xi) - g(x+\xi)]\,d\xi$$

where g is the inverse cosine transform of G.

Since

$$C[e^{-\alpha x^2}] = \frac{2}{\sqrt{4\pi\alpha}}\, e^{-\frac{\omega^2}{4\alpha}}$$

we need $\dfrac{1}{4\alpha} = kt$ or $\alpha = \dfrac{1}{4kt}$, so

$$C[e^{-\frac{x^2}{4kt}}] = \frac{2}{\sqrt{4\pi\frac{1}{4kt}}}\, e^{-kt\omega^2}$$

or

$$\frac{1}{2}\sqrt{\frac{\pi}{kt}}\,C[e^{-\frac{x^2}{4kt}}] \;=\; e^{-kt\omega^2}$$

Therefore

$$g \;=\; \frac{1}{2}\sqrt{\frac{\pi}{kt}}\,e^{-\frac{x^2}{4kt}}$$

Now the first term is

$$\frac{1}{2}\sqrt{\frac{\pi}{kt}}\,\frac{1}{\pi}\,\int_0^\infty f(\xi)\left[e^{-\frac{(x-\xi)^2}{4kt}} \;-\; e^{-\frac{(x+\xi)^2}{4kt}}\right] d\xi$$

The second term is

$$S^{-1}[\frac{2}{\pi\omega}] \;=\; 1$$

The last term is again by convoultion of 1 with the same function g, that is

$$\frac{1}{2}\sqrt{\frac{\pi}{kt}}\,\frac{1}{\pi}\,\int_0^\infty 1\cdot\left[e^{-\frac{(x-\xi)^2}{4kt}} \;-\; e^{-\frac{(x+\xi)^2}{4kt}}\right] d\xi$$

If we decide to use (9.5.15) then remember that the inverse sine transform is giving the constant 1 for $x \geq 0$. Combining all these terms

$$u(x,t) \;=\; \frac{1}{2}\sqrt{\frac{1}{\pi kt}}\,\int_0^\infty f(\xi)\left[e^{-\frac{(x-\xi)^2}{4kt}} \;-\; e^{-\frac{(x+\xi)^2}{4kt}}\right] d\xi + 1 - \frac{1}{2}\sqrt{\frac{1}{\pi kt}}\,\int_0^\infty 1\cdot\left[e^{-\frac{(x-\xi)^2}{4kt}} \;-\; e^{-\frac{(x+\xi)^2}{4kt}}\right] d\xi$$

$$u(x,t) \;=\; \sqrt{\frac{1}{4\pi kt}}\,\int_0^\infty [f(\xi) \;-\; 1]\left[e^{-\frac{(x-\xi)^2}{4kt}} \;-\; e^{-\frac{(x+\xi)^2}{4kt}}\right] d\xi \;+\; 1$$

4. Since the boundary condition is on u_x, we have to use Fourier cosine transform:

$$U_t = -k\omega^2 U \qquad \text{since } u_x(0,t) = 0$$

The solution is

$$U = \underbrace{F(\omega)}_{C[f]}\,\underbrace{e^{-k\omega^2 t}}_{C[g]}$$

This g is exactly the same as in the previous problem.

5. To prove (9.5.14), we start with the definition of inverse Fourier sine transform

$$S^{-1}[H] \,=\, h(x) \,=\, \int_0^\infty S(f)C(g)\sin\omega x\,d\omega$$

Substitute for the Fourier sine transform of f

$$S^{-1}[H] \,=\, h(x) \,=\, \int_0^\infty \left(\frac{2}{\pi}\int_0^\infty f(\xi)\sin\omega\xi\,d\xi\right)\,C(g)\sin\omega x\,d\omega$$

We now rearrange and put the integral over ω inside

$$S^{-1}[H] \,=\, h(x) \,=\, \frac{2}{\pi}\int_0^\infty f(\xi)\int_0^\infty C(g)\sin\omega\xi\,\sin\omega x\,d\omega\,d\xi$$

We can use the trigonometric identity

$$\sin\omega\xi\,\sin\omega x \,=\, \frac{1}{2}\cos\omega(x-\xi) \,-\, \frac{1}{2}\cos\omega(x+\xi)$$

and get two integrals

$$S^{-1}[H] \,=\, h(x) \,=\, \frac{1}{\pi}\int_0^\infty f(\xi)\left[\int_0^\infty C(g)\cos\omega(x-\xi)\,d\omega \,-\, \int_0^\infty C(g)\cos\omega(x+\xi)\,d\omega\right]d\xi$$

Now each of the inner integrals is inverse Fourier cosine transform of g at $x-\xi$ and $x+\xi$. Thus

$$S^{-1}[H] \,=\, h(x) \,=\, \frac{1}{\pi}\int_0^\infty f(\xi)\,[g(x-\xi) \,-\, g(x+\xi)]\,d\xi$$

To prove (9.5.15), we substitute for the inverse Fourier cosine transform of g and go through similar arguments

$$S^{-1}[H] \,=\, h(x) \,=\, \int_0^\infty \left(\frac{2}{\pi}\int_0^\infty g(\xi)\cos\omega\xi\,d\xi\right)\,S(f)\sin\omega x\,d\omega$$

We now rearrange and put the integral over ω inside

$$S^{-1}[H] \,=\, h(x) \,=\, \frac{2}{\pi}\int_0^\infty g(\xi)\int_0^\infty S(f)\cos\omega\xi\,\sin\omega x\,d\omega\,d\xi$$

We can use the trigonometric identity

$$\cos\omega\xi\,\sin\omega x \,=\, \frac{1}{2}\sin\omega(\xi+x) \,-\, \frac{1}{2}\sin\omega(\xi-x)$$

and get two integrals

$$S^{-1}[H] \,=\, h(x) \,=\, \frac{1}{\pi}\int_0^\infty g(\xi)\left[\int_0^\infty S(f)\sin\omega(\xi+x)\,d\omega \,-\, \int_0^\infty S(f)\sin\omega(\xi-x)\,d\omega\right]d\xi$$

Now each of the inner integrals is inverse Fourier sine transform of f at $\xi+x$ and $\xi-x$. Thus

$$S^{-1}[H] \,=\, h(x) \,=\, \frac{1}{\pi}\int_0^\infty g(\xi)\,[f(\xi+x) \,-\, f(\xi-x)]\,d\xi$$

6. To prove (9.5.16), we start with the definition of inverse Fourier cosine transform

$$C^{-1}[H] \;=\; h(x) \;=\; \int_0^\infty C(f)C(g)\cos\omega x\, d\omega$$

Substitute for the Fourier cosine transform of g

$$C^{-1}[H] \;=\; h(x) \;=\; \int_0^\infty \left(\frac{2}{\pi}\int_0^\infty g(\xi)\cos\omega\xi\, d\xi\right) C(f)\cos\omega x\, d\omega$$

We now rearrange and put the integral over ω inside

$$C^{-1}[H] \;=\; h(x) \;=\; \frac{2}{\pi}\int_0^\infty g(\xi)\int_0^\infty C(f)\cos\omega\xi\,\cos\omega x\, d\omega\, d\xi$$

We can use the trigonometric identity

$$\cos\omega\xi\,\cos\omega x \;=\; \frac{1}{2}\cos\omega(x-\xi) \,+\, \frac{1}{2}\cos\omega(x+\xi)$$

and get two integrals

$$C^{-1}[H] \;=\; h(x) \;=\; \frac{1}{\pi}\int_0^\infty g(\xi)\left[\int_0^\infty C(f)\cos\omega(x-\xi)\, d\omega \,+\, \int_0^\infty C(f)\cos\omega(x+\xi)\, d\omega\right] d\xi$$

Now each of the inner integrals is inverse Fourier cosine transform of f at $x-\xi$ and $x+\xi$. Thus

$$C^{-1}[H] \;=\; h(x) \;=\; \frac{1}{\pi}\int_0^\infty g(\xi)\,[f(x-\xi) \,-\, f(x+\xi)]\, d\xi$$

7.

a. By definition

$$S(1) = \frac{2}{\pi}\int_0^\infty \sin \omega x dx$$
$$= \frac{2}{\pi}\frac{-1}{\omega}\cos \omega x = \frac{2}{\pi\omega}.$$

b. The Fourier cosine transform $C\left(\int_0^x \phi(t)dt\right)$ can be derived by using

$$\phi(x) = \frac{df}{dx}$$

and (9.5.6)

$$S\left(\frac{df}{dx}\right) = -\omega C(f).$$

Divide by $-\omega$ we get

$$C\left(\int_0^x \phi(t)dt\right) = -\frac{1}{\omega}S(\phi)$$

c. As in part c. the Fourier sine transform $S\left(\int_0^x \phi(t)dt\right)$ can be derived by using

$$\phi(x) = \frac{df}{dx}$$

and (9.5.5)

$$C\left(\frac{df}{dx}\right) = -\frac{2}{\pi}f(0) + \omega S(f)$$

for

$$f(x) = \int_0^x \phi(t)dt$$

which satisfies $f(0) = 0$. Divide by ω we get

$$S\left(\int_0^x \phi(t)dt\right) = \frac{1}{\omega}C(\phi)$$

8. Note that

$$\frac{1}{\omega}e^{-\omega\alpha} = \int_{\alpha}^{\infty} e^{-\omega t}dt.$$

Therefore

$$\begin{aligned}
S^{-1}\left(\frac{1}{\omega}e^{-\omega\alpha}\right) &= S^{-1}\left(\int_{\alpha}^{\infty} e^{-\omega t}dt\right) \\
&= \int_{\alpha}^{\infty} \frac{x}{x^2+t^2}dt \\
&= \int_{\alpha}^{\infty} \frac{1}{x(1+(t/x)^2}dt \\
&= \frac{1}{x}\arctan(t/x)\Big|_{t=\alpha}^{t\to\infty} \\
&= \frac{\pi}{2x} - \frac{1}{x}\arctan\left(\frac{\alpha}{x}\right).
\end{aligned}$$

9.6 Fourier Transform in 2 Dimensions

Problems

1. Solve the wave equation

$$
\begin{aligned}
&u_{tt} = c^2 \nabla^2 u, \qquad -\infty < x < \infty\,, \qquad -\infty < y < \infty, \\
&u(x, y, 0) = f(x, y), \\
&u_t(x, y, 0) = 0.
\end{aligned}
$$

1. Fourier transform in two dimension of the given wave equation yields:

$$U_{tt} \;=\; c^2\left(-\,|\vec{\omega}|^2\right)U \;=\; -\,c^2|\vec{\omega}|^2\,U(\omega_1,\omega_2,t)$$

The solution is

$$U(\omega_1,\omega_2,t) \;=\; A(\vec{\omega})\;\cos c|\vec{\omega}|t \;+\; B(\vec{\omega})\;\sin c|\vec{\omega}|t$$

Using the Initial conditions in the transform domain, we get

$$U(\omega_1,\omega_2,t) \;=\; F(\vec{\omega})\;\underbrace{\cos c|\vec{\omega}|t}_{G(\vec{\omega})}$$

By the convolution theorem

$$u(x,t) \;=\; \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vec{r_0})\,g(\vec{r}\,-\,\vec{r_0})\,d\vec{r_0}$$

We only need to find

$$g(\vec{r}) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos c|\vec{\omega}|t\,e^{-i\vec{\omega}\cdot\vec{r}}\,d\vec{\omega}$$

10 Green's Functions

10.1 Introduction

10.2 One Dimensional Heat Equation

Problems

1. Consider the heat equation in one dimension

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q(x,t), \qquad 0 < x < 1, \qquad t > 0,$$
$$u(x,0) = f(x),$$
$$u(0,t) = A(t),$$
$$u(1,t) = B(t).$$

Obtain a solution in the form (10.2.16).

2. Consider the same problem subject to the homogeneous boundary conditions

$$u_x(0,t) = u_x(1,t) = 0$$

a. Obtain a solution by any method.

b. Obtain a solution in the form (10.2.16).

3. Solve the wave equation in one dimension

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + Q(x,t), \qquad 0 < x < 1, \qquad t > 0,$$
$$u(x,0) = f(x),$$
$$u_t(x,0) = g(x),$$
$$u(0,t) = 0,$$
$$u(1,t) = 0.$$

Define functions such that a solution in a similar form to (10.2.16) exists.

4. Solve the above wave equation subject to

a.
$$u_x(0,t) = u_x(1,t) = 0.$$

b.
$$u_x(0,t) = 0, \qquad u_x(1,t) = B(t).$$

c.
$$u(0,t) = A(t), \qquad u_x(1,t) = 0.$$

1. $$u_t = u_{xx} + Q(x,\,t) \quad 0 \le x \le 1 \quad t > 0$$

$$u(x,\,0) = f(x)$$
$$u(0,\,t) = A(t)$$
$$u(1,\,t) = B(t)$$

Solution: Let $w(x,\,t) = A(t) + x[B(t) - A(t)]$ and let

$$v(x,\,t) = u(x,\,t) - w(x,\,t)$$

Then $$v_t = v_{xx} + y(x,\,t) \quad \text{where} \quad y(x,\,t) = Q(x,\,t) - w_t + w_{xx}$$

$$v(x,\,0) = g(x) \equiv f(x) - A(0) - x[B(0) - A(0)]$$

$$v(0,\,t) = v(1,\,t) = 0$$

The Homogeneous solution has eigenfunctions and eigenvalues

$$\Phi_n(x) = \sin(n\,\pi\,x), \quad \lambda_n = (n\,\pi)^2, \qquad n = 1,\,2,\,\ldots$$

$$y(x,\,t) = \sum_{n=1}^{\infty} y_n(t)\phi_n(x)$$

where

$$y_n(t) = \frac{\int_0^1 y(x,\,t)\sin(n\,\pi\,x)\,dx}{\int_0^1 \sin^2(n\,\pi\,x)\,dx} = 2\int_0^1 y(x,\,t)\sin(n\,\pi\,x)\,dx$$

Let $$v(x,\,t) = \sum_{n=1}^{\infty} v_n(t)\phi_n(x) \quad \text{then} \quad v(x,\,0) = g(x) = \sum_{n=1}^{\infty} v_n(0)\phi_n(x)$$

So $$v_n(0) = \frac{\int_0^1 g(x)\sin(n\,\pi\,x)\,dx}{\int_0^1 \sin^2(n\,\pi\,x)\,dx} = 2\int_0^1 g(x)\sin(n\,\pi\,x)\,dx$$

Substitute in the equation:

$$\sum_{n=1}^{\infty} v_n'(t)\sin(n\,\pi\,x) = \sum_{n=1}^{\infty}(-(n\,\pi)^2)v_n(t)\sin(n\,\pi\,x) + \sum_{n=1}^{\infty} y_n(t)\sin(n\,\pi\,x)$$

or $$\sum_{n=1}^{\infty}\Big[v_n'(t) + (n\,\pi)^2 v_n(t) - y_n(t)\Big]\sin(n\,\pi\,x) = 0$$

so $$v_n'(t) + (n\,\pi)^2 v_n(t) = y_n(t)$$

where $$v_n(0) = 2\int_0^1 g(x)\sin n\,\pi\,x\,dx$$

This has a solution $$v_n(t) = v_n(0)e^{-(n\,\pi)^2 t} + \int_0^t y_n(\tau)e^{-(n\,\pi)^2(t-\tau)}d\tau$$

(using variation of parameters). Thus

$$v(x,\,t) = \sum_{n=1}^{\infty} \sin(n\,\pi\,x)\left[e^{-(n\,\pi)^2 t}\;2\int_0^1 g(s)\sin(n\,\pi\,s)ds + \int_0^t 2\int_0^1 y(s,\,\tau)\sin(n\,\pi\,s)ds\;e^{-(n\,\pi)^2(t-\tau)}\right]d\tau$$

$$= \int_0^1 g(s)\left[2\sum_{n=1}^{\infty}\sin(n\,\pi\,x)e^{-(n\,\pi)^2 t}\sin(n\,\pi\,s)\right]ds$$

$$+ \int_0^1\int_0^t y(s,\,\tau)\left[2\sum_{n=1}^{\infty}\sin(n\,\pi\,x)e^{-(n\,\pi)^2(t-\tau)}\sin(n\,\pi\,s)\right]d\tau ds$$

so
$$v(x,\,t) = \int_0^1 [f(x) - A(0) - x\,[B(0) - A(0)]\,]\,G(x;\,s,\,t)ds$$

$$+ \int_0^1\int_0^t [Q(x,\,t) - A'(t) + x\,[B'(t) - A'(t)]\,]\,G(x;\,s,\,t-\tau)d\tau ds$$

where
$$G(x;\,s,\,t) = 2\sum_{n=1}^{\infty}\sin(n\,\pi\,x)e^{-(n\,\pi)^2 t}\sin(n\,\pi\,s)$$

2. $$u_t = u_{xx} + Q(x,\, t) \quad 0 \le x \le 1 \quad t > 0$$
$$u(x,\, 0) = f(x)$$
$$u_x(0,\, t) = u_x(1,\, t) = 0$$

(a). The homogeneous solution has eigenfunctions and eigenvalues

$$\Phi_n(x) = \cos(n\,\pi\, x) \quad \lambda_n = (n\,\pi)^2 \quad n = 0,\, 1,\, 2,\, \ldots$$

so $$u(x,\, t) = \sum_{n=0}^{\infty} a_n(t) \cos(n\,\pi\, x)$$

$$a_n(0) = \frac{\int_0^1 f(x) \cos(n\,\pi\, x)\, dx}{\int_0^1 \cos^2(n\,\pi\, x)\, dv} = 2 \int_0^1 f(x) \cos(n\,\pi\, x)\, dx$$

Expanding $$Q(x,\, t) = \sum_{n=0}^{\infty} q_n(t) \cos(n\,\pi\, x)$$
where the coefficients

$$q_n(t) = 2 \int_0^1 Q(x,\, t) \cos(n\,\pi\, x)\, dx$$

so $$a_n(t) = a_n(0) e^{-(n\,\pi)^2 t} + e^{-(n\,\pi)^2 t} \int_0^t q_n(\tau) e^{(n\,\pi)^2 \tau} d\tau$$

Thus
$$u(x,\, t) = \sum_{n=0}^{\infty} \cos(n\,\pi\, x) \left[2 \int_0^1 f(s) \cos(n\,\pi\, s) ds\, e^{-(n\,\pi)^2 t} + \right.$$
$$\left. e^{-(n\,\pi)^2 t} \int_0^t \int_0^1 2Q(s,\, \tau) \cos(n\,\pi\, s) ds\, e^{(n\,\pi)^2 \tau} d\tau \right]$$

$$= \int_0^1 f(s) \left[\sum_{n=0}^{\infty} 2 \cos(n\,\pi\, x) \cos(n\,\pi\, s) e^{-(n\,\pi)^2 t} \right] ds +$$

$$\int_0^1 \int_0^t Q(s,\, \tau) \left[\sum_{n=0}^{\infty} 2 \cos(n\,\pi\, x) \cos(n\,\pi\, s) e^{-(n\,\pi)^2 (t-\tau)} \right] d\tau ds$$

(b). $$u(x,\, t) = \int_0^1 f(s) G(x;\, s,\, t) ds + \int_0^1 \int_0^t Q(s,\, \tau) G(x;\, s,\, t-\tau) d\tau ds$$

where $$G(x;\, s,\, t) = \sum_{n=0}^{\infty} 2 \cos(n\,\pi\, x) \cos(n\,\pi\, s) e^{-(n\,\pi)^2 t}$$

3. $u_{tt} = u_{xx} + Q(x,\, t) \quad 0 \le x \le 1 \quad t > 0$

$u(x,\, 0) = f(x)$
$u_t(x,\, 0) = g(x)$
$u(0,\, t) = u(1,\, t) = 0$

The homgeonous solution has eigenfunctions and eigenvalues

$$Q_n(x) = \sin(n\,\pi\,x), \quad \lambda_n = (n\pi)^2 \quad n = 1,\, 2, \ldots$$

$$Q(x,\, t) = \sum_{n=1}^{\infty} q_n(t)\sin(n\pi x) \quad \text{so} \quad q_n(t) = 2\int_0^1 Q(x,\, t)\sin(n\,\pi\,x)\,dx$$

Let $$u(x,\, t) = \sum_{n=1}^{\infty} A_n(t)\sin(n\pi x)$$

Then $$f(x) = \sum_{n=1}^{\infty} A_n(0)\sin(n\pi x) \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} A_n'(0)\sin(n\pi x)$$

So $$A_n''(t) + (n\,\pi)^2 A_n(t) = q_n(t) \quad \text{where}$$

$$A_n(0) = 2\int_0^1 f(x)\sin(n\pi x)\,dx \quad \text{and} \quad A_n'(0) = 2\int_0^1 g(x)\sin(n\pi x)\,dx$$

The solution is then

$$A_n(t) = K_{1n}\cos(n\pi t) + K_{2n}\sin(n\pi t) + [A_n(t)]_p$$

where the particular solution is

$$[A_n(t)]_p = \frac{-\cos(n\pi t)}{n\pi}\int_0^t \sin(n\pi\tau)q_n(\tau)d\tau + \frac{\sin(n\pi t)}{n\pi}\int_0^t \cos(n\pi\tau)q_n(\tau)d\tau$$

$$K_{1n} = 2\int_0^1 f(s)\sin(n\pi s)ds \quad \text{and} \quad K_{2n} = \tfrac{2}{n\pi}\int_0^1 g(s)\sin(n\pi s)ds$$

So $$u(x,\, t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left[2\int_0^1 f(s)\sin(n\pi s)ds\cos(n\pi t) + \frac{2}{n\pi}\int_0^1 g(s)\sin(n\pi s)ds\sin(n\pi t)\right.$$

$$\frac{-\cos(n\pi t)}{n\pi}\int_0^t \sin(n\pi\tau)2\int_0^1 Q(s,\, \tau)\sin(n\pi s)dsd\tau$$

$$\left.+\frac{\sin(n\pi t)}{n\pi}\int_0^t \cos(n\pi\tau)2\int_0^1 Q(s,\, \tau)\sin(n\pi s)dsd\tau\right\}$$

Rearrange:

$$u(x,\, t) = \int_0^1 f(s)\left[\sum_{n=1}^{\infty} 2\sin(n\pi x)\sin(n\pi s)\cos(n\pi t)\right] ds$$

$$+\int_0^1 g(s)\left[\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi x)\sin(n\pi s)\sin(n\pi t)\right]ds$$

$$+\int_0^1\int_0^t Q(s,\tau)\left[\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi x)\sin(n\pi s)(-\cos(n\pi t))\sin(n\pi\tau)\right]d\tau ds$$

$$+\int_0^1\int_0^t Q(s,\tau)\left[\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi x)\sin(n\pi s)\sin(n\pi t)\cos(n\pi\tau)\right]d\tau ds$$

$$u(x,\, t)=\int_0^1 f(s)\left[\sum_{n=1}^{\infty}2\sin(n\pi x)\sin(n\pi s)\cos(n\pi t)\right]ds$$

$$+\int_0^1 g(s)\left[\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi x)\sin(n\pi s)\sin(n\pi t)\right]ds$$

$$+\int_0^1\int_0^t Q(s,\tau)\left[\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi x)\sin(n\pi s)\left\{\sin(n\pi t)\cos(n\pi\tau)-\cos(n\pi t)\sin(n\pi\tau)\right\}\right]d\tau ds$$

But $\quad\{\sin(n\pi t)\cos(n\pi t)-\cos(n\pi t)\sin(n\pi\tau)\}=\sin(n\pi(t-\tau))$

So Let $\quad G(x;\, s,\, t)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi x)\sin(n\pi s)\sin(n\pi t)$

Then $\quad u(x,\, t)=\int_0^1 f(s)G_t(x;\, s,\, t)ds+\int_0^1 g(s)G(x;\, s,\, t)ds$

$$+\int_0^1\int_0^t Q(s,\, \tau)G(x;\, s,\, t-\tau)d\tau ds$$

4 a. $$u_{tt} = u_{xx} + Q(x,\, t) \quad 0 \le x \le 1 \quad t > 0$$

$$u(x,\, 0) = f(t)$$
$$u_t(x,\, 0) = g(x)$$
$$u_x(0,\, t) = u_x(1,\, t) = 0$$

For the homogenous problem, the eigenfunctions and eigenvalues are

$$\Phi_n(x) = \cos(n\,\pi\,x), \quad \lambda_n = (n\,\pi)^2 \quad n = 0,\, 1,\, 2, \ldots$$

Expand in terms of the eigenfunctions:

$$Q(x,\, t) = \sum_{n=1}^{\infty} q_n(t)\cos(n\,\pi\,x)$$

where $$q_n(t) = 2\int_0^1 Q(x,\, t)\cos(n\,\pi\,x)\,dx \qquad n = 1,\, 2, \ldots$$

Let $$u(x,\, t) = \sum_{n=0}^{\infty} A_n(t)\cos(n\,\pi\,x).$$

For $n = 0$:

$$A_0(0) = \int_0^1 f(x)\,dx$$

$$A_0'(0) = \int_0^1 g(x)\,dx$$

$$q_0(t) = \int_0^1 Q(x,t)\,dx$$

$$A_0''(t) = q_0(t) = \int_0^1 Q(x,t)\,dv$$

$$A_0(t) = \frac{t^2}{2}\int_0^1 Q(x,t)\,dx + c_1 t + c_2$$

$$A_0(0) = \int_0^1 f(x)\,dx = c_2$$

$$A_0'(0) = \int_0^1 g(x)\,dx = c_1$$

$$A_0(t) = \frac{t^2}{2}\int_0^1 Q(s,t)ds + t\int_0^1 f(s)\,ds + \int_0^1 g(s)\,ds$$

For $n > 0$:

$$u(x,\, 0) = f(x) = \sum_{n=1}^{\infty} A_n(0)\cos(n\,\pi\,x)$$

where $$A_n(0) = 2\textstyle\int_0^1 f(x)\cos(n\pi x)\,dx$$

$$u_t(x,\,0) = g(x) = \sum_{n=1}^{\infty} A_n'(0)\cos(n\,\pi\,x)$$

where $$A_n'(0) = 2\int_0^1 g(x)\cos(n\pi x)\,dx$$

$$A_n''(t) + (n\pi)^2 A_n(t) = q_n(t)$$

$$A_n(t) = k_{1n}\cos(n\pi t) + k_{2n}\sin(n\pi t) + [A_n(t)]_p$$

$$[A_n(t)]_p = \frac{-\cos(n\pi t)}{n\pi}\int_0^t \sin(n\pi\tau)q_n(\tau)d\tau + \frac{\sin(n\pi t)}{n\pi}\int_0^t \cos(n\pi\tau)q_n(\tau)d\tau$$

$$k_{1n} = 2\int_0^1 f(s)\cos(n\pi s)ds$$

$$k_{2n} = \frac{2}{n\pi}\int_0^1 g(s)\sin(n\pi s)ds$$

So $$u(x,t) = \sum_{n=1}^{\infty}\cos(n\,\pi\,x)\left[2\int_0^1 f(s)\cos(n\pi s)ds\cos(n\pi t)\right.$$

$$+\frac{2}{n\pi}\int_0^1 g(s)\cos(n\pi s)ds\sin(n\pi t)$$

$$-\frac{\cos(n\pi t)}{n\pi}\int_0^t \sin(n\pi\tau)2\int_0^1 Q(s,\,\tau)\cos(n\pi s)dsd\tau$$

$$\left.+\frac{\sin(n\pi t)}{n\pi}\int_0^t \cos(n\pi\tau)2\int_0^t Q(s,\tau)\cos(n\pi s)dsd\tau\right] + A_0(t)$$

$$u(x,t) = \int_0^1 f(s)\left[\sum_{n=1}^{\infty} 2\cos(n\,\pi\,x)\cos(n\pi s)\cos(n\pi t)\right]ds$$

$$+\int_0^1 g(s)\left[\sum_{n=1}^{\infty}\frac{2}{n\pi}\cos(n\,\pi\,x)\cos(n\pi s)\sin(n\pi t)\right]ds$$

$$+\int_0^1\int_0^t Q(s,\tau)\left\{\sum_{n=1}^{\infty}\frac{2}{n\pi}\cos(n\,\pi\,x)\cos(n\pi s)\sin(n\pi(t-\tau))\right\}d\tau ds$$

$$+A_0(t)$$

where we have used the identity

$$[\sin(n\pi t)\cos(n\pi\tau) - \cos(n\pi t)\sin(n\pi\tau)] = \sin(n\pi(t-\tau))$$

Let $$G(x;s,\,t) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\cos(n\,\pi\,x)\cos(n\pi s)\sin(n\pi t)$$

Then
$$u(x,\,t) = +\int_0^1 f(s)G_t(x;\,s,\,t)\,ds + \int_0^1 g(s)G(x;\,s,\,t)\,ds$$
$$\int_0^1 \int_0^t Q(s,\,\tau)G(x;\,s,\,t-\tau)d\tau ds + \frac{t^2}{2}\int_0^1 Q(s,\,t)ds + t\int_0^1 f(s)ds + \int_0^1 g(s)ds$$

4b. Same as 4a, but $u_x(0,\, t) = 0 \quad u_x(1,\, t) = \beta(t)$

Let $w(x,\, t) = \dfrac{x^2}{2}\beta(t), \quad so \quad w_x(0,\, t) = 0 \quad w_x(1,\, t) = \beta(t)$

$$v(x,\, t) = u(x,\, t) - w(x,\, t)$$

$$v(x,\, 0) = f(x) - \frac{x^2}{2}\beta(0)$$

$$v_t(x,\, 0) = g(x) - \frac{x^2}{2}\beta'(0)$$

$$v_x(0,\, t) = v_x(1,\, t) = 0$$

$$v_{tt} = v_{xx} + Q(x,\, t) - w_{tt}(x,\, t) + w_{xx}(x,\, t)$$

$$w_{tt}(x,\, t) = \frac{x^2}{2}\beta''(t)$$

$$w_{xx}(x,\, t) = \beta(t)$$

So $u(x,\, t) = v(x,\, t) + w(x,\, t)$ where $v(x,\, t)$ is as in 4a.

with $\hat{f}(x) = f(x) - \dfrac{x^2}{2}\beta(0), \quad \hat{g}(x) = g(x) - \dfrac{x^2}{2}\beta'(0)$

and $\hat{Q}(x,\, t) = Q(x,\, t) - \dfrac{x^2}{2}\beta''(t) + \beta(t)$

$$u(x,\, t) = \frac{x^2}{2}\beta(t) + \frac{t^2}{2}\int_0^1 \left(Q(s,\, t) - \frac{s^2}{2}\beta''(t) + \beta(t)\right) ds$$

$$+ \, t\int_0^1 (f(s) - \frac{s^2}{2}\beta(0))ds + \int_0^1 (g(s) - \frac{s^2}{2}\beta'(0))ds$$

$$+ \int_0^1 (f(s) - \frac{s^2}{2}\beta(0))G_t(x;\, s,\, t)ds + \int_0^1 (g(s) - \frac{s^2}{2}\beta'(0))G(x;\, s,\, t)ds$$

$$+ \int_0^1 \int_0^t \left[Q(s,\, \tau)) - \frac{s^2}{2}\beta''(\tau) + \beta(\tau)\right] G(x;\, s,\, t-\tau)d\tau ds$$

where $G(x;\, s,\, t-\tau) = \displaystyle\sum_{n=1}^{\infty} \frac{2}{n\pi} \cos(n\pi x)\cos(n\pi s)\sin(n\pi(t-\tau))$

4c. $\quad u(0,\, t) = A(t) \quad u_x(1,\, t) = 0$

Let $\quad w(x,\, t) = (1-x)^2\, A(t)$

Then $\quad w(0,\, t) = A(t) \quad w_x(1,\, t) = -2(1-x)A(t)|_{x=1} = 0$

As before letting $\quad v(x,\, t) = v(x,\, t) - w(x,\, t)$

$$v_{tt} = v_{xx} + \hat{Q}(x,\, t)$$

where $\quad \hat{Q}(x,\, t) = Q(x,\, t) + (1-x)^2 A'' + 2A(t)$

$$v(x,\, 0) = f(x) - (1-x)^2 A(0) \equiv \hat{f}(x)$$

$$v_t(x,\, 0) = g(x) - (1-x)^2 A'(0) \equiv \hat{g}(x)$$

$$v(0,\, t) = v_x(1,\, t) = 0$$

The homogenous equation has eigenfunctions and eigenvalues

$$\phi_n(x) = \cos\left(n - \frac{1}{2}\right)\pi x \quad \lambda_n = \left[\left(n - \frac{1}{2}\right)\pi\right]^2 \quad n = 1,\, 2,\, \ldots$$

$$\hat{Q}(x,\, t) = \sum_{n=1}^{\infty} q_n(t) \cos\left[\left(n - \frac{1}{2}\right)\pi x\right]$$

where $\quad q_n(t) = 2\int_0^1 \hat{Q}(x,\, t)\cos\left[\left(n - \frac{1}{2}\right)\pi x\right]\, dx$

$$v(x,\, t) = \sum_{n=1}^{\infty} A_n(t) \cos\left[\left(n - \frac{1}{2}\right)\pi x\right]$$

$$v(x,\, 0) = \sum_{n=1}^{\infty} A_n(0) \cos\left[\left(n - \frac{1}{2}\right)\pi x\right] = \hat{f}(x)$$

where $\quad A_n(0) = \dfrac{\int_0^1 \hat{f}(x)\cos(n - \frac{1}{2})\pi x\, dx}{\int_0^1 \cos^2(n - \frac{1}{2})\pi x\, dx}$

$$A_n(0) = 2\int_0^1 \hat{f}(x)\cos\left[\left(n - \frac{1}{2}\right)\pi x\right]\, dx$$

$$v_t(x,\, 0) = \sum_{n=1}^{\infty} A_n'(0) \cos\left[\left(n - \frac{1}{2}\right)\pi x\right] = \hat{g}(x)$$

where $\quad A_n'(0) = 2\int_0^1 \hat{g}(x)\cos\left(n - \frac{1}{2}\right)\pi x\, dx$

$$A_n''(t) + \left[\left(n - \frac{1}{2}\right)\pi\right]^2 A_n(t) = q_n(t)$$

The solution is

$$A_n(t) = K_{1n} \cos\left[\left(n - \frac{1}{2}\right)\pi t\right] + K_{2n} \sin\left[\left(n - \frac{1}{2}\right)\pi t\right] + [A_n(t)]_p$$

where the particular solution is

$$[A_n(t)]_p = \frac{-\cos\left[\left(n-\frac{1}{2}\right)\pi t\right]}{(n-\frac{1}{2})\pi} \int_0^t \sin\left[\left(n - \frac{1}{2}\right)\pi\tau\right] q_n(\tau) d\tau$$

$$+\frac{\sin\left[\left(n-\frac{1}{2}\right)\pi t\right]}{(n-\frac{1}{2})\pi} \int_0^t \cos\left[\left(n - \frac{1}{2}\right)\pi\tau\right] q_n(\tau) d\tau$$

$$K_{1n} = 2\int_0^1 \hat{f}(x) \cos\left[\left(n - \frac{1}{2}\right)\pi x\right] dx$$

$$K_{2n} = \frac{2}{(n-\frac{1}{2})\pi} \int_0^1 \hat{g}(x) \cos\left[\left(n - \frac{1}{2}\right)\pi x\right] dx$$

$$u(x,\, t) = v(x,\, t) + w(x,\, t)$$

$$= \sum_{n=1}^{\infty} A_n(t) \cos\left[\left(n - \frac{1}{2}\right)\pi x\right] + (1-x)^2 A(t)$$

$$= (1-x)^2 A(t) + \sum_{n=1}^{\infty} \cos\left[\left(n - \frac{1}{2}\right)\pi x\right] \left[2\int_0^1 \hat{f}(s)\, \cos\left[\left(n - \frac{1}{2}\right)\pi s\right] ds \cos\left[(n - \frac{1}{2})\pi t\right]\right.$$

$$+\frac{2}{(n-\frac{1}{2})\pi} \int_0^1 \hat{g}(s) \cos\left[\left(n - \frac{1}{2}\right)\pi s\right] ds \sin\left[\left(n - \frac{1}{2}\right)\pi t\right]$$

$$+2\int_0^1 \int_0^t \hat{Q}(s,\, \tau) \cos\left[\left(n - \frac{1}{2}\right)\pi s\right] \left\{\frac{\sin\left[(n-\frac{1}{2})\pi t\right]}{(n-\frac{1}{2})\pi} \cos\left[\left(n - \frac{1}{2}\right)\pi\tau\right]\right.$$

$$\left. - \frac{\cos\left[\left(n-\frac{1}{2}\right)\pi t\right]}{(n-\frac{1}{2})\pi} \sin\left[\left(n - \frac{1}{2}\right)\pi\tau\right]\right\} d\tau ds$$

Let $$G(x;\, s,\, t) = \sum_{n=1}^{\infty} \frac{2}{(n-\frac{1}{2})\pi} \cos\left[\left(n - \frac{1}{2}\right)\pi x\right] \cos\left[\left(n - \frac{1}{2}\right)\pi s\right]$$

Then $$u(x,\, t) = (1-x)^2 A(t) + \int_0^1 \left[f(s) - (1-s)^2 A(0)\right] G_t(x;\, s,\, t) ds$$

$$+\int_0^1 \left[g(s) - (1-s)^2 A'(0)\right] G(x;\, s,\, t) ds$$

$$+\int_0^1 \int_0^t \left[Q(s,\, \tau) + (1-s)^2 A''(\tau) + 2A(\tau)\right] G(x;\, s,\, t-\tau) d\tau ds$$

10.3 Green's Function for Sturm-Liouville Problems

Problems

1. Show that Green's function is unique if it exists.
Hint: Show that if there are 2 Green's functions $G(x;s)$ and $H(x;s)$ then

$$\int_0^1 [G(x;s) - H(x;s)]\, f(s) ds = 0.$$

2. Find Green's function for each

a.

$$\begin{aligned} -k u_{xx} &= f(x), \qquad 0 < x < L, \\ u'(0) &= 0, \\ u(L) &= 0. \end{aligned}$$

b.

$$\begin{aligned} -u_{xx} &= F(x), \qquad 0 < x < L, \\ u'(0) &= 0, \\ u'(L) &= 0. \end{aligned}$$

c.

$$\begin{aligned} -u_{xx} &= f(x), \qquad 0 < x < L, \\ u(0) - u'(0) &= 0, \\ u(L) &= 0. \end{aligned}$$

3. Find Green's function for

$$-k y'' + \ell y = 0, \qquad 0 < x < 1,$$

$$y(0) - y'(0) = 0,$$

$$y(1) = 0.$$

4. Find Green's function for the initial value problem

$$\mathcal{L} y = f(x),$$

$$y(0) = y'(0) = 0.$$

Show that the solution is

$$y(x) = \int_0^x G(x;s) f(s) ds.$$

5. Prove (10.3.22).

1. Assume there exist 2 Green's functions $G(x,\, s),\ H(x,\, s)$

Then $$\int_0^1 G(x,\, s) f(s) ds = \int_0^1 H(x,\, s) f(s) ds \quad \text{for all } f.$$

So $$\int_0^1 [G(x,\, s) - H(x,\, s)]\, f(s) ds = 0 \quad \text{for all } f.$$

$\Rightarrow$ $$G(x,\, s) - H(x,\, s) = 0 \Rightarrow G(x,\, s) = H(x,\, s)$$

$\Rightarrow$ G if it exists, is unique.

2a. $$-ku_{xx} = f(x) \qquad 0 < x < L \qquad p(x) = k$$

$$u'(0) = 0 \qquad u(L) = 0$$

Solve: $$-ku'' = 0 \qquad u'(0) = 0$$

$$-kv'' = 0 \qquad v(L) = 0$$

Choose $$c = -1 \qquad \text{so} \quad kW = -1$$

$$u = ax + b \qquad u'(0) = a = 0 \qquad \Rightarrow u = b$$

$$v = \alpha x + \beta \qquad v(L) = 0 = \alpha L + \beta \qquad \beta = -\alpha L \qquad \Rightarrow v = \alpha(x - L)$$

$$W = \begin{vmatrix} b & \alpha(x-L) \\ 0 & \alpha \end{vmatrix} = \alpha b$$

$$k\alpha b = -1$$

Let $b = 1$, then $\alpha = -\dfrac{1}{k}$

Then $\boxed{u(x) = 1}$

and $\boxed{v(x) = -\dfrac{1}{k}x + \dfrac{L}{k} = -\dfrac{1}{k}(x - L)}$

$$G(x,\, s) = \begin{cases} u(s)\, v(x) & 0 \le s \le x \le 1 \\[2ex] u(x)\, v(s) & 0 \le x \le s \le 1 \end{cases}$$

$$G(x,\, s) = \begin{cases} -\dfrac{1}{k}(x - L) & 0 \le s \le x \le 1 \\[2ex] -\dfrac{1}{k}(s - L) & 0 \le x \le s \le 1 \end{cases}$$

2b. $$-v_{xx} = f(x) \quad 0 < x < L$$

$$u'(0) = 0 \quad u'(L) = 0 \quad p(x) = -1$$

$$u''(x) = 0 \quad u'(0) = 0 \Rightarrow u = a$$

$$v''(x) = 0 \quad v'(L) = 0 \Rightarrow v = \alpha$$

$$W = \begin{vmatrix} a & \alpha \\ 0 & 0 \end{vmatrix} = 0 = c$$

So

$$G(x,\, s) = \begin{cases} a\alpha & 0 \le s \le x \le 1 \\ \alpha a & 0 \le x \le s \le 1 \end{cases}$$

Therefore Green's function is a constant

(For a Newman problem, the solution is not unique).

2c. $$-u_{xx} = f(x) \qquad 0 < x < L$$

$$u(0) - u'(0) = 0 \quad u(L) = 0$$

$$-u'' = 0 \quad u = ax + b \quad \Rightarrow \quad (a(0) + b) - a = 0 \quad \Rightarrow \quad b = a$$

$$\Rightarrow \quad \boxed{u = ax + a}$$

$$-v'' = 0 \quad v = \alpha x + \beta \quad v(L) = 0 \quad \Rightarrow \quad \alpha L = -\beta \quad \Rightarrow \quad \beta = -\alpha L$$

$$\Rightarrow \quad \boxed{v = \alpha(x - L)}$$

Let $\quad c = -1$

$$W = \begin{vmatrix} ax + a & \alpha(x - L) \\ a & \alpha \end{vmatrix} = 1 \qquad \Rightarrow \quad \alpha(ax + a) - \alpha a(x - L) = 1$$

Let $$a = 1 \Rightarrow \alpha x + \alpha - \alpha x + \alpha L = 1 \quad \text{or} \quad \alpha(1 + L) = 1 \quad \Rightarrow \quad \alpha = \frac{1}{1 + L}$$

$$\boxed{u = x + 1}$$

$$\boxed{v = \frac{1}{1 + L}(x - L)}$$

$$G(x,\, s) = \begin{cases} (s - 1)(x - L)\,/\,(1 + L) & 0 \le s \le x \le 1 \\ (x - 1)(s - L)\,/\,(1 + L) & 0 \le x \le s \le 1 \end{cases}$$

3. $$-ky'' + \ell y = 0 \qquad 0 < x < 1$$

$$y(0) - y'(0) = 0 \quad y(1) = 0$$

$$-ku'' + \ell u = 0$$

$$u'' - \frac{\ell}{k}u = 0$$

Let $\lambda^2 = \ell/k$ then $u = ae^{\lambda x} + be^{-\lambda x}$

$$u(0) - u'(0) = 0 \quad ae^{\lambda x} + be^{-\lambda x} - a^{\lambda}e^{\lambda x} + b^{\lambda}e^{-\lambda x}\,|_{x=0} = 0$$

$$a + b - a\lambda + b\lambda = 0 \quad \Rightarrow \quad a(1-\lambda) + b(1+\lambda) = 0$$

$$b = a\frac{(-1+\lambda)}{(1+\lambda)}$$

$$\Rightarrow \qquad \boxed{u = a\left[e^{\lambda x} + \frac{\lambda - 1}{\lambda + 1}e^{-\lambda x}\right]}$$

$$-kv'' + \ell v = 0 \quad v(1) = 0$$

$$v'' - \lambda^2 v = 0 \quad v = \alpha e^{\lambda x} + \beta e^{-\lambda x} \quad v(1) = 0 = \alpha e^{\lambda} + \beta e^{-\lambda}$$

$$\beta = \frac{-\alpha e^{\lambda}}{e^{-\lambda}} = -\alpha e^{2\lambda}$$

$$\boxed{v = \alpha\left[e^{\lambda x} - e^{2\lambda}e^{-\lambda x}\right]}$$

$$W = k = \begin{vmatrix} a\left[e^{\lambda x} + \dfrac{(\lambda-1)}{(\lambda+1)}e^{-\lambda x}\right] & \alpha\left[e^{\lambda x} - e^{2\lambda}e^{-\lambda x}\right] \\ a\left[\lambda e^{\lambda x} + \left(\dfrac{-\lambda^2+\lambda}{\lambda+1}\right)e^{-\lambda x}\right] & \alpha\left[\lambda e^{\lambda x} + \lambda e^{2\lambda}e^{-\lambda_x}\right] \end{vmatrix}$$

$$= \alpha a\left[e^{\lambda x} + \frac{(\lambda-1)}{(\lambda+1)}e^{-\lambda x}\right]\left[\lambda e^{\lambda x} + \lambda e^{2\lambda}e^{-\lambda x}\right] - a\alpha\left[e^{\lambda x} - e^{2\lambda}e^{-\lambda x}\right]\left[\lambda e^{\lambda x} - e^{-\lambda x}\left(\frac{\lambda^2-\lambda}{\lambda+1}\right)\right]$$

$$= \alpha a\left[\lambda e^{2\lambda} + \frac{\lambda^2-\lambda}{\lambda+1} + \lambda e^{2\lambda} + \left(\frac{\lambda^2-\lambda}{\lambda+1}\right)\right]$$

$$= 2\alpha a\left[\lambda e^{2\lambda} + \frac{\lambda^2-\lambda}{\lambda+1}\right]$$

$$\alpha = k/2a\left[\lambda e^{2\lambda} + \frac{\lambda^2 - \lambda}{\lambda + 1}\right]$$

Let $a = 1$

$$G(x;\, s) = \begin{cases} \left(e^{\lambda x} + \dfrac{\lambda - 1}{\lambda + 1} e^{-\lambda x}\right) \quad \left(\dfrac{k}{2(\lambda e^{2\lambda} + \frac{\lambda^2 - \lambda}{\lambda + 1}}\right) & \left(e^{\lambda s} - e^{2\lambda} e^{-\lambda s}\right) \\ & 0 \leq s \leq x \leq 1 \\ \text{(by symetry)} & 0 \leq x \leq s \leq 1 \end{cases}$$

4. Let's take $$\mathcal{L}y = \frac{d}{dx}\left(p\frac{dy}{dx}\right) + q\,y$$

$y(x) = \int_0^x G(x;\,s)f(s)ds$ clearly satisfies $y(0) = 0,$ since, in this case, both limits of integration are zero.

$$y'(x) = G(x;\,x)f(x) + \int_0^x \frac{\partial G(x;\,s)}{\partial x} f(s)ds$$

$y'(0) = 0$ implies $\boxed{G(x;\,x) = 0}$

Differentiate again after multiplying by $p(x)$

$$\frac{d}{dx}\left(p\frac{d}{dx}\right) + q\,y = \frac{d}{dx}\left\{p(x)\int_0^x \frac{\partial G(x;\,s)}{\partial x} f(s)ds\right\}$$

$$+q(x)\int_0^x G(x;\,s)f(s)ds$$

Note that $p(x)$ and $q(x)$ can be put inside the integral on s !

$$= \left.\frac{\partial G(x;\,s)}{\partial x}\right|_{s=x} p(x)f(x)$$

$$+ \int_0^x \left\{\frac{\partial}{\partial x}\left[p(x)\frac{\partial G(x;\,s)}{\partial x}\right]\right\} f(s)ds$$

$$+ \int_0^x \{q(x)G(x;\,s)\}\, f(s)ds$$

Thus $$\mathcal{L}y = \int_0^x \mathcal{L}G f(s)ds + \left[p(x)\left.\frac{\partial G(x_j\,s)}{\partial x}\right|_{s=x}\right] f(x)$$

In order to get $f(x)$ on the right hand side we must have

$\boxed{\mathcal{L}G = 0}$ (will anihilate the integral)

and

$$\boxed{\left.\frac{\partial}{\partial x}G(x;\,s)\right|_{s=x} = \frac{1}{p(x)}}$$

we also need

$\boxed{G(x;\,x) = 0}$ (seen earlier)

Another way to solve problem 4

Let $$Ly \equiv (py')' + qy = f$$

$$y(0) = y'(0) = 0$$

Then $Lv = 0$ has 2 linearly independent solutions v_1 and v_2.

Define $$w(x) = v_1(x)\int_0^x v_2(s)f(s)ds - v_2(x)\int_0^x v_1(s)f(s)ds$$

Then $$w'(x) = v_1'(x)\int_0^x v_2(s)f(s)ds - v_2'(x)\int_0^x v_1(s)f(s)ds$$
$$+ v_1(x)v_2(x)f(x) - v_2(x)v_1(x)f(x)$$

So: $$w'(x) = v_1'(x)\int_0^x v_2(s)f(s)ds - v_2'(x)\int_0^x v_1(s)f(s)ds$$

So $$(pw'(x))' = \frac{d}{dx}\left[p(x)v_1'(x)\right]\int_0^x v_2(s)f(s)ds - \frac{d}{dx}\left[p(x)v_2'(x)\right]\int_0^x v_1(s)f(s)ds$$
$$+ p(x)\left[v_1'(x)v_2(x) - v_2'(x)v_1(x)\right]f(x)$$
$$= -q(x)w(x) + \underbrace{p(x)\left[v_1'(x)v_2(v) - v_2'(v)v_1(x)\right]}_{=c\text{(see problem 5 next)}}f(x)$$

Hence $$(pw')' + qw = cf$$

where $$w(0) = w'(0) = 0$$

So $y = \dfrac{w}{c}$ is a solution to

$$(py')' + qy = f$$

with $y(0) = y'(0) = 0$

Thus $$y(x) = \int_0^x f(s)\frac{v_1(x)v_2(s) - v_2(x)v_1(s)}{p(x)\left[v_1'(x)v_2(x) - v_2'(x)v_1\right]}ds$$

So $$y(x) = \int_0^x f(s)G(x;\, s)ds$$

where

$$G(x;\, s) = \begin{cases} 0 & x < s \\ \dfrac{v_1(x)v_2(s) - v_2(x)v_1(s)}{p(x)\left[v_1'(x)v_2(x) - v_2'(x)v_1(x)\right]} & x > s \end{cases}$$

And we can see that

$$LG = 0 \quad \text{For} \quad x > s$$

$$G(x;\, x) = 0$$

$$G_x(x;\, x) = \frac{1}{p(x)}$$

5. Prove $\qquad p(x)W(x) = c$

u, v both satisfythe same ODE i.e.

$$-(pu')' + qu = 0$$

$$-(pv')' + qv = 0$$

$$W(x) = uv' - vu'$$

To compute $p(u'v - v'u)$ we differentiate the Wronskian $W(x)$

$$\frac{dW}{dx} = u'v' + uv'' - v'u' - vu'' \quad \Rightarrow \quad \boxed{uv'' - vu'' = \frac{dW}{dx}}$$

$$-(pu')' + qu = 0 \quad \Rightarrow \quad -p'u' - pu'' + qu = 0$$

also $-p'v' - pv'' + qv = 0$

Divide by p $\qquad \dfrac{-p'}{p}u' - u'' + \dfrac{qv}{p} = 0 \quad OR \quad u'' = \dfrac{-p'}{p}u' + \dfrac{qu}{p}$

$$\frac{-p'}{p}v' - v'' + \frac{qv}{p} = 0 \quad OR \quad v'' = \frac{-p'}{p}v' + \frac{qv}{p}$$

Thus $\qquad uv'' - vu'' = u(\dfrac{-p'}{p}v' + \dfrac{qv}{p}) - v(\dfrac{-p'}{p}u' + \dfrac{qu}{p}) = \dfrac{p'}{p}\,[uv' - vu']$

$$\Rightarrow \frac{dW}{dx} = \frac{-p'}{p}W \qquad \text{so} \quad W'p = Wp'$$

$(Wp)' = W'p + Wp'$ so using the above $(Wp)' = W'p - W'p = 0$

$(Wp)' = 0$ and $Wp = c$

10.4 Dirac Delta Function

Problems

1. Derive (10.4.3) from (10.4.2).

2. Show that (10.4.8) satisfies (10.4.7).

3. Derive (10.4.9)
Hint: use a change of variables $\xi = c(x - x_i)$.

1. $$f(x) = \int_a^b f(x_i)\delta(x - x_i)\,dx \qquad (10.4.2)$$

Let $\quad f(x) \equiv 1 \quad x\epsilon(-\infty,\, \infty). \quad$ Then $\quad f(x_i) = 1 \qquad$ for all x_i

So $$f(x) = 1 = \int_{-\infty}^{\infty} 1 \cdot \delta(x - x_i)\,dx$$

2. Show (10.4.8) satisfies (10.4.7)

$$H(x - x_i) = \int_{-\infty}^{x} \delta(\xi - x_i) d\xi \qquad (10.4.8)$$

$$H(x - x_i) = \begin{cases} 0 & x < x_i \\ \\ 1 & x > x_i \end{cases} \qquad (10.4.7)$$

Let $F(\xi,\, x)$ be defined by

$$F(\xi, x) = \begin{cases} 1 & \xi < x \\ \\ 0 & \xi > x \end{cases}$$

Then $$\int_{-\infty}^{x} \delta(\xi - x_i) d\xi = \int_{-\infty}^{\infty} F(\xi,\, x)\delta(\xi - x_i) d\xi$$

since on the other interval $F \equiv 0$.

But using (10.4.5), the integral on the right is $F(x_i,\, x)$

Notice that

$$F(x_i,\, x) = \begin{cases} 1 & x_i < x \\ \\ 0 & x_i > x \end{cases}$$

which is the definition of the Heaviside function

$$H(x - x_i)$$

Thus we get (10.4.8)

3. Prove: $\delta\left[c(x-x_i)\right] = \dfrac{1}{|\,c\,|}\delta(x-x_i)$

Compute the intergral: $\displaystyle\int_{-\infty}^{\infty} f(x)\delta\left[c(x-x_0)\right]\,dx =$

make a transformation: $y = c(x-x_0)$

then $dx = \dfrac{1}{c}dy$

for $c>0$ the limits: $\displaystyle\int_{-\infty}^{\infty} f\left(\frac{y}{c}+x_0\right)\delta(y)\frac{1}{c}dy \;=\; \frac{1}{c}f(x_0)$

for $c<0$ the limits: $\displaystyle\int_{\infty}^{-\infty} f\left(\frac{y}{c}+x_0\right)\delta(y)\frac{1}{c}\,dy \;=\; -\frac{1}{c}f(x_0)$

Combining the two: $\displaystyle\int_{-\infty}^{\infty} f(x)\delta\left[c\left(x-x_0\right)\right]\,dx \;=\; \frac{1}{|c|}f(x_0)$

But $\displaystyle f(x_0) = \int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,dx$

therefore $\displaystyle\int_{-\infty}^{\infty} f(x)\delta\left[c(x-x_0)\right]\,dx = \frac{1}{|c|}\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,dx$

and we have the required relationship for δ functions.

10.5 Nonhomogeneous Boundary Conditions

Problems

1. Consider
$$u_t = u_{xx} + Q(x,t), \qquad 0 < x < 1, \qquad t > 0,$$
subject to
$$u(0,t) = u_x(1,t) = 0,$$
$$u(x,0) = f(x).$$

a. Solve by the method of eigenfunction expansion.

b. Determine the Green's function.

c. If $Q(x,t) = Q(x)$, independent of t, take the limit as $t \to \infty$ of part (b) in order to determine the Green's function for the steady state.

2. Consider
$$u_{xx} + u = f(x), \qquad 0 < x < 1,$$
$$u(0) = u(1) = 0.$$
Determine the Green's function.

3. Give the solution of the following problems in terms of the Green's function

a. $u_{xx} = f(x)$, subject to $u(0) = A$, $u_x(1) = B$.

b. $u_{xx} + u = f(x)$, subject to $u(0) = A$, $u(1) = B$.

c. $u_{xx} = f(x)$, subject to $u(0) = A$, $u_x(1) + u(1) = 0$.

4. Solve
$$\frac{dG}{dx} = \delta(x - s),$$
$$G(0; s) = 0.$$
Show that $G(x; s)$ is <u>not</u> symmetric.

5. Solve
$$u_{xxxx} = f(x),$$
$$u(0) = u(1) = u_x(0) = u_{xx}(1) = 0,$$
by obtaining Green's function.

1. $$u_t = u_{xx} + Q(x,\, t) \quad 0 \,\le\, x \,\le\, 1 \quad t \,\ge\, 0$$

subject to: $$u(0,\, t) = u_x(1,\, t) = 0 \quad u(x,\, 0) = f(x)$$

a. $$\phi_n(x) = \sin\left(n - \frac{1}{2}\right)\pi x, \qquad n = 1,\, 2, \ldots$$

$$\lambda_n = \left[\left(n - \frac{1}{2}\right)\pi\right]^2, \qquad n = 1,\, 2, \ldots$$

Expand: $$Q(x,\, t) = \sum_{n=1}^{\infty} q_n(t) \sin\left(n - \frac{1}{2}\right)\pi x$$

The coefficients are:

$$q_n(t) = 2\int_0^1 Q(x,\, t) \sin\left(n - \frac{1}{2}\right)\pi x \, dx$$

Expand: $$u(x,\, t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(n - \frac{1}{2}\right)\pi x$$

It was shown in Chapter 8 that the coefficients are:

$$u_n(t) = e^{\left(n - \frac{1}{2}\right)^2 \pi^2 t}\left(2\int_0^1 f(x) \sin\left(n - \frac{1}{2}\right)\pi x \, dx\right) + \int_0^t q_n(\tau) e^{-\left(n - \frac{1}{2}\right)^2 \pi^2 (t-\tau)} \, d\tau$$

So: $$u(x,\, t) = \int_0^1 f(s)\left[\sum_{n=1}^{\infty} 2\sin\left[\left(n - \frac{1}{2}\right)\pi s\right] \sin\left[\left(n - \frac{1}{2}\right)\pi x\right] e^{-\left(n - \frac{1}{2}\right)^2 \pi^2 t}\right] ds$$

$$+ \int_0^1 \int_0^t Q(s,\, \tau)\left[\sum_{n=1}^{\infty} 2\sin\left[\left(n - \frac{1}{2}\right)\pi s\right] \sin\left[\left(n - \frac{1}{2}\right)\pi x\right] e^{-\left(n - \frac{1}{2}\right)^2 \pi^2 (t-\tau)}\right] d\tau ds$$

b. $$G(x;\, s,\, t) = \sum_{n=1}^{n} 2\sin\left[\left(n - \frac{1}{2}\right)\pi s\right] \sin\left[\left(n - \frac{1}{2}\right)\pi x\right] e^{-\left(n - \frac{1}{2}\right)^2 \pi^2 t}$$

c. If $\qquad Q(x,\,t) = Q(x), \qquad$ find the steady state solution

$$\lim_{t\to\infty} u(x,\,t) = \int_0^1 f(s)\left[\sum_{n=1}^{\infty} 2\sin\left(n-\frac{1}{2}\right)\pi s \sin\left(n-\frac{1}{2}\right)\pi x\right] ds \underbrace{\left[\lim_{t\to\infty} e^{-\left(n-\frac{1}{2}\right)^2\pi^2 t}\right]}_{=0}$$

$$+\int_0^1 Q(s)\sum_{n=1}^{\infty} 2\sin\left(n-\frac{1}{2}\right)\pi s \sin\left(n-\frac{1}{2}\right)\pi x\, ds \underbrace{\lim_{t\to\infty} \underbrace{e^{-\left(n-\frac{1}{2}\right)^2\pi^2 t} \underbrace{\int_0^t e^{-\left(n-\frac{1}{2}\right)^2\pi^2 \tau} d\tau}_{\frac{e^{-\left(n-\frac{1}{2}\right)^2\pi^2 t}-1}{\left(n-\frac{1}{2}\right)^2\pi^2}}}_{\frac{1-e^{-\left(n-\frac{1}{2}\right)^2\pi^2 t}}{\left(n-\frac{1}{2}\right)^2\pi^2}}}_{\frac{1}{\left(n-\frac{1}{2}\right)^2\pi^2}}$$

$$= \int_0^1 Q(s)\sum_{n=1}^{\infty} 2\sin\left(n-\frac{1}{2}\right)\pi s \frac{\sin\left(n-\frac{1}{2}\right)\pi x}{\left(n-\frac{1}{2}\right)^2\pi^2}\, ds$$

Therefore

$$\lim_{t\to\infty} u(x,\,t) = \int_0^1 Q(s)\left\{\sum_{n=1}^{\infty} \frac{2}{\left(n-\frac{1}{2}\right)^2\pi^2} \sin\left(n-\frac{1}{2}\right)\pi s \sin\left(n-\frac{1}{2}\right)\pi x\right\} ds$$

2. $$u_{xx} + u = f(x) \qquad 0 < x < 1$$

subject to $$u(0) = u(1) = 0$$

Solve the 2 ODEs to get G:

$$\left.\begin{array}{l} u'' = -u \\ \\ u(0) = 0 \end{array}\right\} \Rightarrow u(x) = a\sin x + b\cos x \qquad u(0) = 0 \Rightarrow b = 0$$

$$\left.\begin{array}{l} v'' = -v \\ \\ v(1) = 0 \end{array}\right\} \Rightarrow v(x) = \alpha\sin x + \beta\cos x \qquad v(1) = 0 \quad \Rightarrow \quad \alpha\sin 1 + \beta\cos 1 = 0 \quad \Rightarrow \beta = -\alpha\tan 1$$

$$u = a\sin x$$

$$v = \alpha\sin x - \alpha\tan 1\cos x$$

$$W = \begin{vmatrix} a\sin x & \alpha(\sin x - \tan 1\cos x) \\ \\ a\cos x & \alpha(\cos x + \tan 1\sin x) \end{vmatrix} = a\alpha\left[\sin x\cos x + \tan 1\sin^2 x - \sin x\cos x + \tan 1\cos^2 x\right]$$

$$W = a\alpha\tan 1$$

So $$a\alpha\tan 1 = c \quad \text{let} \quad c = \tan 1 \Rightarrow a = \frac{1}{\alpha}$$

$$u = \frac{1}{\alpha}\sin x \qquad v = \alpha(\sin x - \tan 1\cos x)$$

$$G(x;\, s) = \begin{cases} \dfrac{1}{\alpha}\sin s\,\alpha(\sin x - \tan 1\cos x) & s \le x \\ \\ \dfrac{1}{\alpha}\sin x\,\alpha(\sin s - \tan 1\cos s) & x \le s \end{cases}$$

Therefore:

$$G(x;\, s) = \begin{cases} \sin s\left[\sin x - \tan 1\cos x\right] & 0 \le s \le x \le 1 \\ \\ \sin x\left[\sin s - \tan 1\cos s\right] & 0 \le x \le s \le 1 \end{cases}$$

3 a. $$u_{xx} = f(x)$$
$$u(0) = A \quad u_x(1) = B$$

To get $G(x;\, s)$,

solve the homogeneous equation with homogeneous boundary conditions

$$u_{xx} = 0 \quad u(0) = 0 \quad u_x(1) = 0$$

$$u'' = 0 \quad u(0) = 0 \Rightarrow \quad u = ax + b \quad b = 0 \qquad \Rightarrow \quad u = ax$$

$$v'' = 0 \quad v'(0) = 0 \Rightarrow \quad v = \alpha x + \beta \quad v'(1) = \alpha = 0 \qquad \Rightarrow \quad v = \beta$$

$$W = \begin{vmatrix} ax & \beta \\ a & 0 \end{vmatrix} = \underbrace{+1}_{\text{coefficent of } u_{xx}} \qquad \Rightarrow -a\beta = 1 \qquad \Rightarrow a = -1/\beta$$

Let $\beta = 1$ then $a = -1$

So $u = -x \quad v = 1$

and

$$G(x;\, s) = \begin{cases} -s & s \le x \\ -x & x \le s \end{cases}$$

$$u(s) = \int_0^1 G(x;\, s) f(x)\, dx - G(1;\, s)B - A\frac{dG(x;\, s)}{dx}\,|_{x=0}$$

$$G(1;\, s) = -s$$

$$\frac{dG}{dx} = \begin{cases} 0 & s \le x \\ -1 & x \le s \end{cases} \quad \text{so} \quad \frac{dG}{dx}\,|_{x=0} = -1$$

$$u(s) = \int_0^1 G(x;\, s) f(x)\, dx + A + Bs$$

$$\text{where} \quad G(x;\, s) = \begin{cases} -s & s \le x \\ \\ -x & x \le s \end{cases}$$

Check the answer by substituting: $\qquad u(s) = \int_0^1 G(x;\, s) f(x)\, dx + A + Bs$

Write this as 2 integrals substituting for G:

$$u(s) = \int_0^s -x f(x)\, dx + \int_s^1 -s f(x)\, dx + A + Bs$$

$$u(0) = \underbrace{\int_0^0 -x f(x)\, dx}_{=0} + \int_0^1 -0 \cdot f(x)\, dx + A + B \cdot 0 = +A \qquad \text{checks}$$

Recall how to differentiate an integral whose limits depend on the variable of integration

$$u'(s) = \underbrace{-s f(s) + s f(s)}_{=0} + \int_s^1 -f(x)\, dx + B$$

Defferentiate again: $\qquad u''(s) = f(s) \qquad$ thus the equation checks

$u'(1) = B \qquad$ the second boundary condition checks

3. b. $$u_{xx} + u = f(x)$$

subject to $$u(0) = A \quad u(1) = B$$

From problem 2,

$$G(x;\, s) = \begin{cases} \sin s\,[\sin x - \tan 1 \cos x] & s \le x \\ \\ \sin x\,[\sin s - \tan 1 \cos s] & x \le s \end{cases}$$

If $\quad x \ge s:$ $$\frac{dG(x;\, s)}{dx} = \sin s \cos x + \sin s \tan 1 \sin x$$

So at $\quad x = 1,$ $$\frac{dG}{dx}\,|_{x=1} = \sin s \cos 1 + \sin s \tan 1 \sin 1$$

If $\quad x \le s;$ $$\frac{dG(x;\, s)}{dx} = \cos x\,[\sin s - \tan 1 \cos s]$$

So at $\quad x = 0,$ $$\frac{dG}{dx}\,|_{x=0} = \sin s - \tan 1 \cos s$$

$$u(s) = \int_0^1 G(x;\, s) f(x) dx + B\,[\sin s(\cos 1 + \tan 1 \sin 1)] - A\,[\sin s - \tan 1 \cos s]$$

3. c.
$$u_{xx} = f(x)$$
$$u(0) = A \quad u_x(1) + u(1) = 0$$

The homogeneous equation with homogeneous boundary conditions:

$$u_{xx} = 0 \quad u(0) = 0 \quad u_x(1) + u(1) = 0$$

$$u'' = 0 \quad u(0) = 0 \qquad \Rightarrow \quad u = ax + b; \quad b = 0 \qquad \Rightarrow \quad u = ax$$

$$v'' = 0 \quad v'(1) + v(1) = 0 \Rightarrow \quad v = \alpha x + \beta; \quad \alpha + \alpha + \beta = 0$$

$$2\alpha + \beta = 0; \Rightarrow \quad \beta = -2\alpha \qquad \Rightarrow \quad v = \alpha x - 2\alpha$$

$$W = \begin{vmatrix} ax & \alpha x - 2\alpha \\ a & \alpha \end{vmatrix} = -2\alpha a = c = -1 \qquad \Rightarrow \quad \alpha a = \frac{1}{2}$$

Let $\quad a = 1 \quad \Rightarrow \quad \alpha = \frac{1}{2}$

Therefore
$$u(x) = x; \qquad v(x) = \frac{1}{2}(x-2)$$

$$G(x;\, s) = \begin{cases} \dfrac{1}{2}s(x-2) & s \le x \\ \dfrac{1}{2}x(s-2) & x \le s \end{cases}$$

If $\quad x \ge s \qquad \dfrac{dG(x,\, s)}{dx} = \dfrac{1}{2}s \qquad \Rightarrow \quad \left.\dfrac{dG}{dx}\right|_{x=1} = \dfrac{1}{2}s \qquad \left.\dfrac{dG}{dx}\right|_{x=0} = \dfrac{s-2}{2}$

$$u(s) = \int_0^1 G(x;\, s) f(x)\, dx - uG_x\Big|_0^1 + Gu_x\Big|_0^1$$

$$uG_x\Big|_0^1 = u(1)\frac{s}{2} - \underbrace{u(0)}_{A}\frac{(s-2)}{2}$$

$$Gu_x\Big|_0^1 = G\,|_{x=1}\ u_x(1) - \underbrace{G\,|_{x=0}}_{=0}\ u_x(0) = -\frac{s}{2}u_x(1)$$

$$\Rightarrow \qquad u(s) = \int_0^1 G(x;\, s) f(x)\, dx - \frac{s}{2}u(1) + A\frac{(s-2)}{2} - \frac{s}{2}u_x(1)$$

$$= \int_0^1 G(x;\, s) f(x)\, dx + A\frac{(s-2)}{2}u(1) - \frac{s}{2}\underbrace{(u(1) + u_x(1))}_{=0}$$

$$u(s) = \int_0^1 G(x;\, s) f(x)\, dx + \frac{1}{2}A(s-2)$$

4. Solve $\quad \dfrac{dG}{dx} = \delta(x-s) \quad$ subject to $\quad G(0;\, s) = 0$

Since $\quad \dfrac{d}{dx}(H(x-s)) = \delta(x-s) \quad \Rightarrow \quad G = H(x-s)$

$$G(0;\, s) = H(-s)$$

$$G(x;\, s) = \begin{cases} 0 & x \geq s \\ \\ 1 & x \leq s \end{cases}$$

Therefore G is <u>NOT</u> symmetric.

5. Solve $$u_{xxxx} = f(x)$$

subject to

$$u(0) = u(1) = u'(0) = u''(1) = 0$$

Since $$\mathcal{L}u = u_{xxxx}$$

with the stated boundary conditions is self-adjoint, we have

$$\int_0^1 (u\mathcal{L}G - G\mathcal{L}u)\, dx = 0$$

for u, G satisfying the boundary conditions.

In particular, if

$$\mathcal{L}u = f, \qquad \mathcal{L}G(x;\, s) = \delta(x - s)$$

then

$$\int_0^1 u\mathcal{L}G\, dx - \int_0^1 G\mathcal{L}u\, dx = \int_0^1 u\delta(x - s)\, dx - \int_0^1 G(s;\, x)f(x)\, dx$$

$$= u(s) - \int_0^1 G(s;\, x)f(x)\, dx.$$

Therefore $$u(x) = \int_0^1 G(x;\, s)f(s)\, ds.$$

Hence we seek a solution to $$\mathcal{L}G(x;\, s) = \delta(x - s)$$

$$G(0;\, s) = G(1;\, s) = G_x(0;\, s) = G_{xx}(1;\, s) = 0$$

Also $$\int_{s^-}^{s^+} G_{xxxx}(x;\, s)\, dx = \int_{s^-}^{s^+} \delta(x - s)\, dx = 1$$

$$\Rightarrow G_{xxx}\Big|_{s^-}^{s^+} = 1.$$

Furthermore, $G_{xx},\ G_x,$ and G are continuous at $x = s$

Using $\mathcal{L}G = 0$ if $x \neq s$,

we can define $G(x;\, s)$ on $[0,\, s)$ and $(s,\, 1]$, and use the above conditions to determine the parameters as follows:

$$G(x;\, s) = \begin{cases} ax^3 + bx^2 + cx + d & x < s \\ \alpha x^3 + \beta x^2 + \gamma x + \delta & x > s \end{cases}$$

$$G(0;\, s) = 0 \qquad \Rightarrow d = 0$$

$$G_x(0;\, s) = 0 \qquad \Rightarrow c = 0$$

$$G(1;\, s) = 0 \qquad \Rightarrow \alpha + \beta + \gamma + \delta = 0$$

$$G_{xx}(1;\, s) = 0 \qquad \Rightarrow 6\alpha + 2\beta = 0 \qquad \Rightarrow \beta = -3\alpha$$

$$\Rightarrow \alpha - 3\alpha + \gamma + \delta = 0$$

$$\Rightarrow \delta = 2\alpha - \gamma$$

$$G(x;\, s) = \begin{cases} ax^3 + bx^2 \\ \alpha x^3 - \beta x^2 + \gamma x + 2\alpha - \gamma \end{cases}$$

$$G_{xx}\Big|_{s^-}^{s^+} = 1 \qquad \Rightarrow 6\alpha - 6a = 1 \qquad \Rightarrow a = \alpha - \frac{1}{6}$$

$$G(x;\, s) = \begin{cases} \left(\alpha - \frac{1}{6}\right) x^3 + bx^2 \\ \alpha x^3 - 3\alpha x^2 + \gamma x + 2\alpha - \gamma \end{cases}$$

$$G_{xx}\,|_{s^+} = G_{xx}\,|_{s^-}$$

$$6\left(\alpha - \frac{1}{6}\right) s + 2b = 6\alpha s - 6\alpha$$

$$-s + 2b = -6\alpha \qquad \Rightarrow b = \frac{s}{2} - 3\alpha$$

$$G(x;\, s) = \begin{cases} \left(\alpha - \frac{1}{6}\right) x^3 + \left(\frac{s}{2} - 3\alpha\right) x^2 \\ \alpha x^3 - 3\alpha x^2 + \gamma x + 2\alpha - \gamma \end{cases}$$

$$G_x\,|_{s^+} = G_x\,|_{s^-}$$

$$3(\alpha - \frac{1}{6})s^2 + (s - 6\alpha)s = 3\alpha s^2 - 6\alpha s + \gamma$$

$$-\frac{1}{2}s^2 + s^2 - 6\alpha s = -6\alpha s + \gamma$$

$$\gamma = \frac{1}{2}s^2$$

$$G(x;\, s) = \begin{cases} (\alpha - \frac{1}{6})x^3 + (\frac{s}{2} - 3\alpha)x^2 \\ \\ \alpha x^3 - 3\alpha x^2 + \frac{s^2}{2}x + 2\alpha - \frac{s^2}{2} \end{cases}$$

$$G\,|_{s^+} = G\,|_{s^-}$$

$$(\alpha - \frac{1}{6})s^3 + (\frac{s}{2} - 3\alpha)s^2 = \alpha s^3 - 3\alpha s^2 + \frac{s^3}{2} + 2\alpha - \frac{s^2}{2}$$

$$\alpha s^3 - \frac{s^3}{6} + \frac{s^3}{2} - 3\alpha s^2 = \alpha s^3 - 3\alpha s^2 + \frac{s^3}{2} + 2\alpha - \frac{s^2}{2}$$

$$2\alpha = \frac{s^2}{2} - \frac{s^3}{6} \quad \Rightarrow \alpha = \frac{s^2}{4} - \frac{s^3}{12}$$

$$G(x;\, s) = \begin{cases} x^3\left[-\frac{s^3}{12} + \frac{s^2}{4} - \frac{1}{6}\right] + x^2\left[\frac{s^3}{4} - \frac{3}{4}s^2 + \frac{s}{2}\right] & x < s \\ \\ x^3\left[-\frac{s^3}{12} + \frac{s^2}{4}\right] + x^2\left[\frac{s^3}{4} - \frac{3}{4}s^2\right] + x\left[\frac{s^2}{2}\right] + \left[\frac{s^2}{2} - \frac{s^3}{6}\right] - \frac{s^2}{2} & x > s \end{cases}$$

$$G(x,\, s) = \begin{cases} x^3\left[-\frac{s^3}{12} + \frac{s^2}{4} - \frac{1}{6}\right] + x^2\left[\frac{s^3}{4} - \frac{3}{4}s^2 + \frac{s}{2}\right] & x < s \\ \\ s^3\left[-\frac{x^3}{12} + \frac{x^2}{4} - \frac{1}{6}\right] + s^2\left[\frac{x^3}{4} - \frac{3}{4}x^2 + \frac{x}{2}\right] & x > s \end{cases}$$

10.6 Fredholm Alternative And Modified Green's Functions

Problems

1. Use Fredholm alternative to find out if

$$u_{xx} + u = \beta + x, \qquad 0 < x < \pi,$$

subject to

$$u(0) = u(\pi) = 0,$$

has a solution for all β or only for certain values of β.

2. Without determining $u(x)$, how many solutions are there of

$$u_{xx} + \gamma u = \cos x$$

a. $\gamma = 1$ and $u(0) = u(\pi) = 0$.

b. $\gamma = 1$ and $u_x(0) = u_x(\pi) = 0$.

c. $\gamma = -1$ and $u(0) = u(\pi) = 0$.

d. $\gamma = 2$ and $u(0) = u(\pi) = 0$.

3. Are there any values of β for which there are solutions of

$$u_{xx} + u = \beta + x, \qquad -\pi < x < \pi$$

$$u(-\pi) = u(\pi),$$
$$u_x(-\pi) = u_x(\pi)?$$

4. Consider

$$u_{xx} + u = 1$$

a. Find the general solution.

b. Obtain the solution satisfying

$$u(0) = u(\pi) = 0.$$

Is your answer consistent with Fredholm alternative?

c. Obtain the solution satisfying

$$u_x(0) = u_x(\pi) = 0.$$

Is your answer consistent with Fredholm alternative?

5. Obtain the solution for

$$u_{xx} - u = e^x,$$

$$u(0) = 0, \qquad u'(1) = 0.$$

6. Determine the modified Green's function required for

$$u_{xx} + u = F(x),$$

$$u(0) = A, \qquad u(\pi) = B.$$

Assume that F satisfies the solvability condition. Obtain the solution in terms of the modified Green's function.

1. $$u_{xx} + u = \beta + x \quad 0 \le x \le \pi$$

$$u(0) = u(\pi) = 0$$

Homogenous: $$u_{xx} + u = 0 \quad u(0) = u(\pi) = 0$$

$$u_H(x) = \sin x \neq 0$$

So $$\int_0^\pi (\beta + x) \sin x \, dx = -\beta \cos x \,|_0^\pi + (\sin x - x \cos x)|_0^\pi$$

$$= -\beta(-1-1) - \pi \cos \pi = +2\beta + \pi = 0 \qquad \Rightarrow \boxed{\beta = -\tfrac{\pi}{2}}$$

2 a. $$u_{xx} + \gamma u = 0 \quad \gamma = 1 \quad u(0) = u(\pi) = 0$$

$$u_H = \sin x$$

$$\int_0^\pi \sin x \cos x \, dx = \frac{1}{2} \sin^2 x \,|_0^\pi = 0$$

$$\boxed{\text{infinite number of solutions}}$$

2 b. $$u_x(0) = u_x(\pi) = 0$$

Has $$u_H = \cos x$$

$$\int_0^\pi \cos^2 x \, dx = \frac{\pi}{2} \neq 0$$

$$\boxed{\text{no solution}}$$

2 c. $$u_{xx} = u \qquad u(0) = u(\pi) = 0$$

$$u_H = e^x$$

No nontrivial homogenous solution

$$\boxed{\text{unique solution}}$$

2 d. $$u_{xx} = -2u$$

$$u_H = \sin(\lambda x) \qquad \Rightarrow \lambda^2 = +2 \qquad \Rightarrow \lambda = \pm\sqrt{2}$$

$$u_H(0) = \sin(\sqrt{2} \cdot 0) = 0 \qquad \text{but} \quad u_H(\pi) = \sin(\sqrt{2} \cdot \pi) \neq 0 \Rightarrow \quad \text{no nontrivial solution}$$

$$\boxed{\text{unique solution}}$$

3. $$u_{xx} + u = \beta + x \quad x \epsilon [-\pi, \pi]$$

$$u(-\pi) = u(\pi)$$

$$u_x(-\pi) = u_x(\pi)$$

$$v_{xx} = -v \quad v_H = \cos x + \sin x$$

$$\int_{-\pi}^{\pi} (\beta + x)(\cos x + \sin x)\, dx = 0$$

Note that integrating an odd function on the symmetric interval gives zero. So we have left:

$$\int_{-\pi}^{\pi} \beta \cos x\, dx + \int_{-\pi}^{\pi} x \sin x\, dx = \beta \sin x \Big|_{-\pi}^{\pi} + (\sin x - x \cos x) \Big|_{-\pi}^{\pi}$$

$$= \beta(0 - 0) + (0 - 0) + (0 - 0) - \pi \cos \pi - \pi \cos(-\pi) = +2\pi \neq 0$$

Regardless of value of $\beta \quad < f, \cos x > \neq 0 \qquad \Rightarrow \boxed{\text{no solution.}}$

4 a. $$u_{xx} + u = 1$$

$$\boxed{\text{Homogenous: } u_{xx} + u = 0 \qquad \Rightarrow u(x) = c_1 \cos x + c_1 \sin x}$$

Get a particular solution of the inhomogeneous

$$u_p = A \quad u_p'' + u_p = 1 \qquad 0 + A = 1 \qquad \Rightarrow A = 1$$

So $$u_g = u_H + u_p = c_1 \cos x + c_2 \sin x + 1$$

4 b. $$u(0) = 0 \qquad \Rightarrow c_1 + 1 = 0$$

$$u(\pi) = 0 \qquad \Rightarrow -c_1 + 1 = 0$$

No solution for the system $\qquad \Rightarrow$ No solution.

$$\int_0^\pi (1) \sin x \, dx = -\cos x \,|_0^\pi = -(-1) + (1) = 2 \neq 0$$

No solution by Fredholm either

4 c. $$u_x(0) = u_x(\pi) = 0$$

$$u_x(0) = c_2 \cos 0 = 0 \quad u_x(\pi) = c_2 \cos \pi = 0$$

$$c_2 = 0 \qquad \Rightarrow \text{ infinite number of solutions}$$

$$\Rightarrow u = c_1 \cos x + 1$$

$$\int_0^\pi 1 \cdot \cos x = \sin x \,|_0^\pi = 0 \qquad \Rightarrow \text{ infinite number of solutions by Fredholm.}$$

5. This problem is similar to the example (10.6.2)-(10.6.3). In this case $A = -1$ which is NOT n^2 for any integer. Thus the nonhomogeneous has a unique solution (Fredholm alternative). The solution can be found by the method of eigenfunction expansion. Note that the eigenfunctions are $\sin\left(n - \frac{1}{2}\right)\pi x, \qquad n = 1, 2, \ldots$. Expandind the right hand side in terms of the eigenfunctions we have

$$e^x = \sum_{n=1}^{\infty} \alpha_n \sin\left(n - \frac{1}{2}\right)\pi x,$$

with coefficients given by

$$\alpha_n = \frac{\int_0^\pi e^x \sin\left(n - \frac{1}{2}\right)\pi x \, dx}{\int_0^\pi \sin^2\left(n - \frac{1}{2}\right)\pi x \, dx}$$

Using the same expansion for u, we get the coefficients u_n by substitution in the differential equation

$$u_n = -\frac{\alpha_n}{1 + \left(n - \frac{1}{2}\right)^2 \pi^2}$$

6. $$u_{xx} + u = F(x) \quad u(0) = A \quad u(\pi) = B$$

Solve the homogenous problem first to find $\hat{G}(x,\, s)$

$$u_{xx} + u = 0 \quad u(0) = 0 = u(\pi)$$

The homogenous problem has $u_H = \sin x$ as a nontrivial solution

(Assume $\int_0^\pi F(x) \sin x \, dx = 0$)

$$\int_0^\pi \sin x \left[\delta(x-s) + c \sin x\right] \, dx = 0 = \sin s + c \underbrace{\int_0^\pi \sin^2 x \, dx}_{\frac{\pi}{2}}$$

$$\Rightarrow c = -\frac{\sin s}{\pi/2}$$

$$\mathcal{L}\hat{G} = \frac{d^2\hat{G}}{dx^2} + \hat{G} = \delta(x-s) + \frac{2 \sin x \sin s}{\pi}$$

$$\hat{G}(0;\, s) = \hat{G}(\pi;\, s) = 0$$

Assume $\hat{G}$ is continuous at $x = s$

$$\int_{s^-}^{s^+} \frac{d^2\hat{G}}{dx^2} \, dx + \int_{s^-}^{s^+} \hat{G} \, dx = \int_{s^-}^{s^+} \delta(x-s) \, dx + \int_{s^-}^{s^+} \frac{2 \sin x \sin s}{\pi} \, dx$$

$$\frac{d\hat{G}}{dx}\Big|_{s^-}^{s^+} + 0 = 1 + 0 \quad \text{(by continuity)}$$

For $x \neq s$ $$\frac{d^2\hat{G}}{dx^2} + \hat{G} = \frac{2 \sin x \sin s}{\pi}$$

Try variation of parameters $$\hat{G}'' + \hat{G} = \left(\frac{2 \sin s}{\pi}\right) \sin x$$

$$\hat{G}_H = c_1 \cos x + c_2 \sin x$$

$$\hat{G}_p(x) = u_1(x) \cos x + u_2(x) \sin x$$

$$\hat{G}'_p(x) = u_1'(x) \cos x - u_1(x) \sin x + u_2'(x) \sin x + u_2(x) \cos x$$

$$= \underbrace{[u_1' \cos x + u_2' \sin x]}_{\text{set this to zero}} + [-u_1 \sin x + u_2 \cos x]$$

$$\hat{G}_p''(x) = -u_1' \sin x + u_2' \cos x - u_1 \cos x - u_2 \sin x$$

$$\hat{G}_p'' + \hat{G}_p = -u_1' \sin x + u_2' \cos x = \frac{2 \sin s}{\pi} \sin x$$

Therefore u_1' and u_2' satisfy the system

$$u_1' \cos x + u_2' \sin x = 0$$

$$-u_1' \sin x + u_2' \cos x = \sin x \left(\frac{2 \sin s}{\pi} \right)$$

The Wronskian is:

$$W = \cos^2 x + \sin^2 x = 1$$

The solution is:

$$u_1' = -\sin x \sin x \left(\frac{2 \sin s}{\pi} \right)$$

$$u_1 = -\frac{2 \sin s}{\pi} \left[\int \sin^2 x \, dx \right] = \frac{2 \sin s}{\pi} \left[-\frac{1}{2} \cos x \sin x + \frac{x}{2} \right]$$

$$u_1 = \frac{1}{\pi} \sin s \sin x \cos x - \frac{x}{\pi} \sin s$$

$$u_2' = \cos x \sin x \left(\frac{2 \sin s}{\pi} \right)$$

$$u_2 = \frac{2 \sin s}{\pi} \int \cos x \sin x \, dx = \frac{2 \sin s}{\pi} \left[\frac{1}{2} \sin^2 x \right] = \frac{1}{\pi} \sin s \sin^2 x$$

The solution is

$$\hat{G}(x;\, s) = c_1 \cos x + c_2 \sin x + \frac{1}{\pi} \cos x \sin s \left[\cos x \sin x - x \right] + \frac{1}{\pi} \sin s \sin^2 x \sin x$$

$$= c_1 \cos x + c_2 \sin x - \frac{x}{\pi} \cos x \sin s + \frac{\sin s}{\pi} \sin x \left[\cos^2 x + \sin^2 x \right]$$

$$= c_1 \cos x + c_2 \sin x - \frac{x}{\pi} \cos x \sin s + \frac{\sin s}{\pi} \sin x$$

So

$$\hat{G}(x;\, s) = \begin{cases} \dfrac{1}{\pi} \sin s \sin x - \dfrac{x}{\pi} \cos x \sin s + c_1 \cos x + c_2 \sin x & x < s \\[2ex] \dfrac{1}{\pi} \sin x \sin s - \dfrac{x}{\pi} \cos x \sin s + c_3 \cos x + c_4 \sin x & x > s \end{cases}$$

From the endpoints: $G(0;\, s) = 0 \Rightarrow c_1 = 0$

$$G(\pi;\, s) = 0 \Rightarrow c_3(-1) + \sin s = 0 \Rightarrow c_3 = +\sin s$$

$$\hat{G}(x;\, s) = \begin{cases} \dfrac{1}{\pi}\sin s \sin x - \dfrac{x}{\pi}\cos x \sin s + c_2 \sin x & x < s \\[2ex] \dfrac{1}{\pi}\sin x \sin s - \dfrac{x}{\pi}\cos x \sin s + \sin s \cos x + c_4 \sin x & x > s \end{cases}$$

Continuity at $\quad x = s; \quad c_2 \sin s = +\sin s \cos s + c_4 \sin s$

$$\Rightarrow c_2 = c_4 + \cos s$$

$$\hat{G}(x;\, s) = \begin{cases} \dfrac{1}{\pi}\sin s \sin x - \dfrac{x}{\pi}\cos x \sin s + \cos s \sin x + c \sin x & x < s \\[2ex] \dfrac{1}{\pi}\sin x \sin s - \dfrac{x}{\pi}\cos x \sin s + \cos x \sin s + c \sin x & x > s \end{cases}$$

Checking jump in derivative

$$\frac{d\hat{G}}{dx}\Big|_{s^-}^{s^+} = 1 \qquad \Rightarrow \cos s \cos x + \sin x \sin s\Big|_{x \to s} = \sin^2(s) + \cos^2(s) = 1$$

Symmetry: Letting $\qquad c = -\dfrac{s}{\pi}\cos s \qquad$ and reordering terms

$$\boxed{\hat{G}(x;\, s) = \begin{cases} \dfrac{1}{\pi}\sin s \sin x + \cos s \sin x - \dfrac{x}{\pi}\cos x \sin s - \dfrac{s}{\pi}\cos s \sin x & x < s \\[2ex] \dfrac{1}{\pi}\sin x \sin s + \cos x \sin s - \dfrac{s}{\pi}\cos s \sin x - \dfrac{x}{\pi}\cos x \sin s & x > s \end{cases}}$$

To obtain a solution using $\hat{G}(x,\, s)$, notice

$$\int_0^\pi \left[u\mathcal{L}\hat{G} - \hat{G}\mathcal{L}u\right]\, dx = \int_0^\pi \left[u(\hat{G}'' + \hat{G}) - \hat{G}(u'' + u)\right]\, dx$$

$$= \int_0^\pi \left[u\hat{G}'' - \hat{G}u'' + (u\hat{G} - u\hat{G})\right] dx = \left(u\hat{G}' - \hat{G}u'\right)\Big|_0^\pi = u\hat{G}'\Big|_0^\pi$$

$$= B\hat{G}'\Big|_\pi - A\hat{G}'\Big|_0$$

$$\frac{d\hat{G}}{dx}\Big|_{\pi} = \left(\frac{1}{\pi}\sin s\cos x - \sin x\sin s - \frac{s}{\pi}\cos s\cos x - \frac{1}{\pi}\cos x\sin s + \frac{x}{\pi}\sin x\sin s\right)\Big|_{x=\pi}$$

$$= -\frac{1}{\pi}\sin s + \frac{s}{\pi}\cos s + \frac{1}{\pi}\sin s = \frac{s}{\pi}\cos s$$

$$\frac{d\hat{G}}{dx}\Big|_{0} = \left(\frac{1}{\pi}\sin s\cos x + \cos s\cos x - \frac{1}{\pi}\cos x\sin s + \frac{x}{\pi}\sin x\sin s - \frac{s}{\pi}\cos s\cos x\right)\Big|_{x=0}$$

$$= \frac{1}{\pi}\sin s + \cos s - \frac{1}{\pi}\sin s - \frac{s}{\pi}\cos s = \left(1 - \frac{s}{\pi}\right)\cos s$$

Also notice:

$$\int_0^{\pi}\left[u\mathcal{L}\hat{G} - \hat{G}\mathcal{L}u\right]\,dx = \int_0^{\pi} u\left[\delta(x-s) + \frac{2}{\pi}\sin s\sin x\right]\,dx - \int_0^{\pi}\hat{G}F\,dx$$

$$= u(s) + \underbrace{\int_0^{\pi} u(x)\frac{2\sin s\sin x}{\pi}\,dx}_{\substack{\text{a multiple of the homogenous solution}\\ \text{so disregard to get particular solution}}} - \int_0^{\pi}\hat{G}F(x)\,dx$$

So $$u(s) - \int_0^{\pi}\hat{G}F(x)\,dx = B\frac{d\hat{G}}{dx}\bigg|_{x=\pi} - A\frac{d\hat{G}}{dx}\bigg|_{x=0}$$

Interchanging x and s, we get

$$\boxed{u(x) = \int_0^{\pi}\hat{G}(x;\,s)F(s)ds + \frac{B}{\pi}x\cos x - A\left(1 - \frac{x}{\pi}\right)\cos x}$$

10.7 Green's Function For Poisson's Equation

Problems

1. Derive Green's function for Poisson's equation on infinite three dimensional space. What is the condition at infinity required to ensure vanishing contribution from the boundary integral?

2. Show that Green's function (10.7.37) satisfies the boundary condition (10.7.35).

3. Use (10.7.39) to obtain the solution of Laplace's equation on the upper half plane subject to

$$u(x,0) = h(x)$$

4. Use the method of eigenfunction expansion to determine $G(\vec{r};\vec{r_0})$ if

$$\nabla^2 G = \delta(\vec{r} - \vec{r_0}), \qquad 0 < x < 1, \quad 0 < y < 1$$

subject to the following boundary conditions

$$G(0,y;\vec{r_0}) = G_x(1,y;\vec{r_0}) = G_y(x,0;\vec{r_0}) = G_y(x,1;\vec{r_0}) = 0$$

5. Solve the above problem inside a unit cube with zero Dirichlet boundary condition on all sides.

6. Derive Green's function for Poisson's equation on a circle by using the method of images.

7. Use the above Green's function to show that Laplace's equation inside a circle of radius ρ with

$$u(r,\theta) = h(\theta) \qquad \text{for } r = \rho$$

is given by Poisson's formula

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\theta_0)\frac{\rho^2 - r^2}{r^2 + \rho^2 - 2\rho r\cos(\theta - \theta_0)}d\theta_0.$$

8. Determine Green's function for the right half plane.

9. Determine Green's function for the upper half plane subject to

$$\frac{\partial G}{\partial y} = 0 \qquad \text{on } y = 0.$$

Use it to solve Poisson's equation

$$\nabla^2 u = f$$

$$\frac{\partial u}{\partial y} = h(x), \qquad \text{on } y = 0.$$

Ignore the contributions at infinity.

10. Use the method of images to solve

$$\nabla^2 G = \delta(\vec{r} - \vec{r_0})$$

in the first quadrant with $G = 0$ on the boundary.

1. Solve

$$\nabla^2 u = f(\vec{r}), \quad \vec{r} = (x,\, y,\, z) \,\epsilon\, \mathcal{R}^3$$

Green's function $G(\vec{r}; \vec{r}_0)$ satisfies

$$\nabla^2 G = \delta(x - x_0)\; \delta(y - y_0)\; \delta(z - z_0)$$

Because of symmetry we have

$$\frac{1}{\rho^2}\,\frac{d}{d\rho}\left(\rho^2\,\frac{dG}{d\rho}\right) = 0 \quad \text{for} \quad \rho \neq 0$$

where $\rho = |\vec{x} - \vec{x}_0|$

The solution is done by integration

$$\rho^2\,\frac{dG}{d\rho} = C$$

Divide by ρ^2 and integrate again

$$G(\rho) = -\frac{C}{\rho} + D$$

To obtain the constants, we integrate over a sphere of radius ρ containing the point $(x_0,\, y_0,\, z_0)$, thus

$$\int\!\!\int\!\!\int \nabla^2\, G\, dx\, dy\, dz = \int\!\!\int\!\!\int \delta(x - x_0)\; \delta(y - y_0)\; \delta(z - z_0) dx\, dy\, dz = 1.$$

But by Green's formula, the left hand side is

$$\int\!\!\int_S \nabla G \cdot n dS = \int\!\!\int_S \frac{\partial G}{\partial \rho}\, dS = \frac{\partial G}{\partial \rho}\, 4\pi\rho^2$$

where S is the surface of the sphere.

Thus

$$\frac{\partial G}{\partial \rho}\, 4\pi\rho^2 = 1$$

$$\frac{\partial G}{\partial \rho} = \frac{1}{4\pi\rho^2}$$

Since $G = -\dfrac{C}{\rho} + D$

then $\dfrac{\partial G}{\partial \rho} = \dfrac{C}{\rho^2}$. Comparing $\dfrac{\partial G}{\partial \rho}$ we have

$$\frac{C}{\rho^2} = \frac{1}{4\pi\rho^2} \Longrightarrow C = \frac{1}{4\pi}$$

We can take $D = 0$, thus

$$G(\rho) = -\frac{1}{4\pi\rho}$$

The condition at infinity can be obtained by using Green's formula as in the text.

$$\lim_{\rho\to\infty} \int\int_S (u\nabla G - G\nabla u) \cdot \vec{n} dS = 0$$

$$\lim_{\rho\to\infty} \rho^2 \left(u\frac{\partial G}{\partial \rho} - G\frac{\partial u}{\partial \rho} \right) = 0$$

or by using G from above:

$$\boxed{\lim_{\rho\to\infty} \left(u + \rho\frac{\partial u}{\partial \rho} \right) = 0}$$

2. Show that

$$G = \frac{1}{2\pi}\ \ln|\vec{r} - \vec{r}_0| - \frac{1}{2\pi}\ \ln|\vec{r} - \vec{r}_0^{\,*}|$$

satisfies

$$G(x,\, 0;\, x_0,\, y_0) = 0$$

Recall

$$\vec{r} = (x,\, y)$$

$$\vec{r}_0 = (x_0,\, y_0)$$

$$\vec{r}_0^{\,*} = (x_0,\, -y_0)$$

$$G(x,\, 0;\, x_0,\, y_0) = \frac{1}{2\pi}\ln\sqrt{(x - x_0)^2 + (0 - y_0)^2}$$

$$-\frac{1}{2\pi}\ln\sqrt{(x - x_0)^2 + (0 + y_0)^2}$$

Since the terms under square root signs are identical we get zero for G.

3. Use

$$u(\vec{r}) = \int\int f(\vec{r}_0)\, G(\vec{r};\vec{r}_0) d\vec{r}_0 + \int_{-\infty}^{\infty} h(x_0) \frac{y/\pi}{(x-x_0)^2+y^2} dx_0$$

To obtain the solution of

$$\nabla^2 u = 0 \quad y > 0$$

$$u(x,\, 0) = h(x).$$

Since this is Laplace's equation $f(\vec{r}) \equiv 0$ and the first integral is zero, thus

$$u(\vec{r}) = \int_{-\infty}^{\infty} h(x_0) \frac{\frac{y}{\pi}}{(x-x_0)^2+y^2} dx_0$$

4. Use the method of eigenfunction expansion to determine $G(\vec{r};\, \vec{r}_0)$ on $[0,\, 1] \times [0,\, 1]$

Because of the boundary conditions, the eigenfunctions are: $\cos n\pi y \quad n = 0,\, 1, \ldots$

$$G(\vec{r};\, \vec{r}_0) = \sum_{n=0}^{\infty} g_n(x) \cos n\pi y$$

or

$$G(\vec{r};\, \vec{r}_0) = \sum_{n=1}^{\infty} \hat{g}_n(y) \cos\left(n + \frac{1}{2}\right)\pi x$$

We take the first expansion and substitute in the equation

$$\sum_{n=0}^{\infty} g_n''(x) \cos n\pi y + \sum_{n=1}^{\infty} (-n^2\pi^2) g_n(x) \cos n\pi y = 0, \quad \vec{r} \neq \vec{r}_0.$$

$$g_n''(x) - n^2\pi^2 g_n(x) = 0 \quad n = 1,\, 2,\, \ldots$$

$$g_0'' \qquad\qquad = 0$$

Subject to:

$$g_n(0) = 0$$

$$g_n'(1) = 0$$

Solve, apply the jump condition for the derivative and so on.

5.

$$\nabla^2 G = \delta(\vec{r} - \vec{r}_0) \quad 0 \le x,\, y,\, z \,\le\, 1$$

$$G(0,\, y,\, z) = G(1,\, y,\, z) = 0$$

$$G(x,\, 0,\, z) = G(x,\, 1,\, z) = 0$$

$$G(x,\, y,\, 0) = G(x,\, y,\, 1) = 0$$

Because of the boundary conditions, the eigenfunctions are: $\sin n\pi y\, \sin m\pi z$, $\; n = 1,\, 2,\, \ldots,$ $m = 1,\, 2,\, \ldots$

Thus

$$G(\vec{r};\, \vec{r}_0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} g_{nm}(x) \sin n\pi y \sin m\pi z$$

(other possibilities exist, depending on the two variables we use. We can even take expansion in all 3 variables !)
Substitute in the equation

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} g''_{nm}(x) \sin n\pi y \sin m\pi z +$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-n^2\pi^2 - m^2\pi^2) g_{nm}(x) \sin n\pi y \sin m\pi z = 0, \quad \vec{r} \ne \vec{r}_0.$$

$$g''_{nm}(x) - (n^2\pi^2 + m^2\pi^2) g_{nm}(x) = 0, \quad n,\, m = 1,\, 2,\, \ldots$$

$$g_{nm}(0) = g_{nm}(1) = 0,$$

The boundary conditions are the result of using the two boundary conditions we didn't use in getting the eigenfunction, in this case $G(0,\, y,\, z) = G(1,\, y,\, z) = 0$.

Solve the boundary value problem to get $g_{nm}(x)$

6. Use the method of images to derive Green's function for Poisson's equation on a circle.

$$\nabla^2 G(\vec{r}; \vec{r}_0) = \delta(\vec{r} - \vec{r}_0) \quad |\vec{r}| \le a$$

$$G(\vec{r}; \vec{r}_0) = 0 \quad \text{for} \quad |\vec{r}| = a$$

Let $\vec{r}_0^*$ be the reflected image of a point $\vec{r}_0$ (see figure). Then if G is the Green's function for the whole plane, we have

$$\nabla^2 G(\vec{r}; \vec{r}_0) = \delta(\vec{r} - \vec{r}_0) - \delta(\vec{r} - \vec{r}_0^*)$$

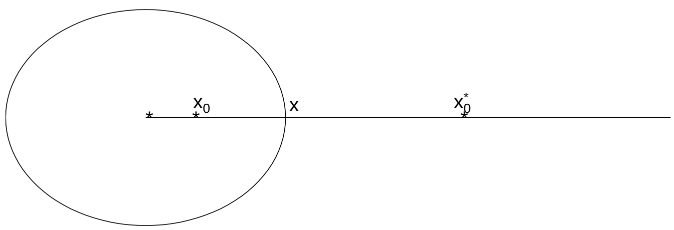


Figure 61: The case $\theta = 0$

So

$$G(\vec{r}; \vec{r}_0) = \frac{1}{2\pi} \ln|\vec{r} - \vec{r}_0| - \frac{1}{2\pi} \ln|\vec{r} - \vec{r}_0^*| + C$$

$$= \frac{1}{4\pi} \ln \frac{|\vec{r} - \vec{r}_0|^2}{|\vec{r} - \vec{r}_0^*|^2} + C$$

To find C, we note that when $|\vec{r}| = a \quad G(\vec{r}; \vec{r}_0) = 0$ and with $\theta = 0$ all 3 points coincide

$$\Longrightarrow C = -\frac{1}{4\pi} \ln \frac{|a - \vec{r}_0|^2}{|a - \vec{r}_0^*|^2}$$

For $\theta \ne 0$ (see figure) $G(\vec{r}; \vec{r}_0) = 0$
yields

$$\frac{1}{4\pi}\left[\ln \frac{|\vec{r} - \vec{r}_0|^2}{|\vec{r} - \vec{r}_0^*|^2} - \ln \frac{|a - \vec{r}_0|^2}{|a - \vec{r}_0^*|^2}\right] = 0$$

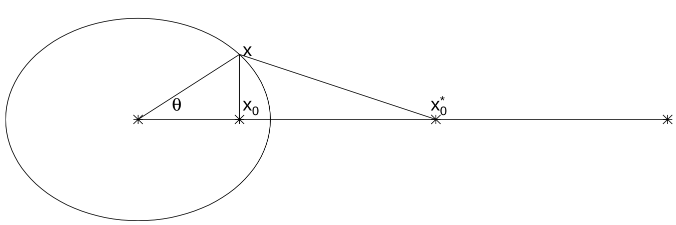


Figure 62: The case $\theta \ne 0$

So for all $|\vec{r}| = a$,

$$\frac{|\vec{r} - \vec{r}_0|^2}{|\vec{r} - \vec{r}_0^*|^2} = \frac{|a - \vec{r}_0|^2}{|a - \vec{r}_0^*|^2}$$

Note

$$|\vec{r} - \vec{r}_0|^2 = |\vec{r}|^2 + |\vec{r}_0|^2 - 2|\vec{r}|\,|\vec{r}_0|\cos\theta$$

$$|\vec{r} - \vec{r}_0^*|^2 = |\vec{r}|^2 + |\vec{r}_0^*|^2 - 2|\vec{r}|\,|\vec{r}_0^*|\cos\theta$$

Therefore

$$|\vec{r}_0^*| = \frac{a^2}{|\vec{r}_0|^2}$$

Note that when $\vec{r}_0$ is the center of the circle that $\vec{r}_0^*$ is at infinity.

Hence

$$G(\vec{r};\,\vec{r}_0) = \frac{1}{4\pi}\ln\left(\frac{|\vec{r} - \vec{r}_0|^2\ a^2}{|\vec{r} - \vec{r}_0^*|^2\ |\vec{r}_0|^2}\right)$$

7. The solution in general is given by

$$u(\vec{r}) = \int\int \underbrace{f(\vec{r}_0)}_{\text{right hand side of equation}=0} G(\vec{r};\, \vec{r}_0) d\vec{r}_0$$

$$+ \oint \underbrace{h(\vec{r}_0)}_{\text{u on the boundary.}} \nabla_{\vec{r}_0} G(\vec{r};\, \vec{r}_0) \cdot \vec{n}\, dS$$

The normal to the circle is in the direction of radius.

$$= \oint h(\vec{r}_0) \frac{\partial}{\partial \vec{r}_0} G(\vec{r};\, \vec{r}_0)|_{r_0=a} dS$$

Convert G obtained in problem 6 to polar coordinates, differentiate and substitute in the integral.

8. Let

$$\vec{r} = (x,\, y)$$

$$\vec{r}_0 = (x_0,\, y_0)$$

Then

$$\vec{r}_0^{\,*} = (-x_0,\, y_0)$$

is the image for the right half plane.

Therefore

$$G(\vec{r};\, \vec{r}_0) = \frac{1}{2\pi}\left[\ln|\vec{r} - \vec{r}_0| - \ln|\vec{r} - \vec{r}_0^{\,*}|\right]$$

in a similar fashion to problem 6.

The solution of Poisson's equation in general is

$$u(\vec{r}) = \int\int f(\vec{r}_0) G(\vec{r};\, \vec{r}_0) d\vec{r}_0 + \oint u(\vec{r}_0) \nabla_{\vec{r}_0} G(\vec{r};\, \vec{r}_0) \cdot \vec{n}\, dS$$

9. For the upper half plane

$$G(\vec{r};\,\vec{r}_0) = \frac{1}{4\pi}\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2}$$

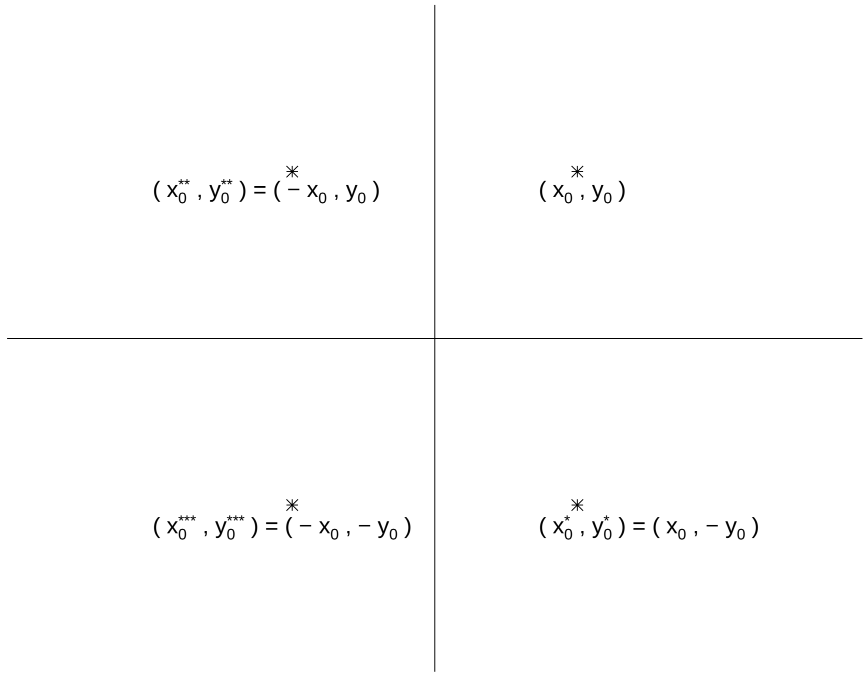
Since $\qquad \vec{r}_0^{\,*} = (x_0,\,-y_0)$

It is straightforward to check that $\dfrac{\partial G}{\partial y} = 0$, when $\quad y = 0.$

The solution is given by

$$u(\vec{r}) = \int\int f(\vec{r}_0)G(\vec{r};\,\vec{r}_0)d\vec{r}_0 + \int_{-\infty}^{\infty} G(x,\,y;\,x_0,\,0)h(x_0)dx_0$$

10. To solve the problem in the first quadrant we take a reflection to the fourth quadrant and the two are reflected to the left half.



$$\nabla^2 G = \delta(\vec{r} - \vec{r}_0) - \delta(\vec{r} - \vec{r}_0^{\,*}) - \delta(\vec{r} - \vec{r}_0^{\,**})$$
$$+ \delta(\vec{r} - \vec{r}_0^{\,***})$$
$$G = \frac{1}{2\pi} \ln \frac{|\vec{r} - \vec{r}_0| \quad |\vec{r} - \vec{r}_0^{\,***}|}{|\vec{r} - \vec{r}_0^{\,*}| \quad |\vec{r} - \vec{r}_0^{\,**}|}$$
$$= \frac{1}{4\pi} \ln \frac{[(x - x_0)^2 + (y - y_0)^2] \quad [(x + x_0)^2 + (y + y_0)^2]}{[(x - x_0)^2 + (y + y_0)^2] \quad [(x + x_0)^2 + (y - y_0)^2]}$$

It is easy to see that on the axes $G = 0$.